Aspects of mass gap, confinement and N=2 structure in 4-D Yang-Mills theory

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We introduce new variables in four dimensional SU(N) Yang-Mills theory. These variables emerge when we sum the path integral over classical solutions and represent the summation as an integral over appropriate degrees of freedom. In this way we get an effective field theory with SU(N)×SU(N) gauge symmetry. In the instanton approximation our effective theory has in addition a N=2 supersymmetry, and when we sum over all possible solutions we find a Parisi-Sourlas supersymmetry. These extra symmetries can then be broken explicitly by a SU(N) invariant and power counting renormalizable mass term. Our results suggest that the confinement mechanism which has been recently identified in the N=2 supersymmetric Yang-Mills theory might also help to understand color confinement in ordinary, pure Yang-Mills theory. In particular, there appears to be an intimate relationship between the N=2 supersymmetry approach to confinement and the Parisi-Sourlas dimensional reduction.

1 Introduction

Recently, there has been impressive progress in understanding confinement in supersymmetric Yang-Mills theories [1]. In particular, the N=2 supersymmetric theory with its supersymmetry properly broken to N=1, appears to materialize the qualitative picture of confinement introduced in [2]. However, despite remarkable success it remains a challenge to extend these results to standard QCD where we still lack a convincing explanation why color and quarks confine.

Intimately related to the problem of color confinement is the origin of a mass gap in Yang-Mills theory. In the case of N=2 supersymmetric Yang-Mills theory a mass gap is introduced explicitly, by adding a mass term to the scalar multiplet [1]. But in the case of ordinary Yang-Mills theory elementary scalar fields are absent. Instead, a mass gap is supposed to have a dynamic origin in the infrared divergence structure of the theory. However, at the moment we still lack a convincing explanation how the mass gap actually appears.

In the present paper we are interested in the problem of color confinement and the related emergence of a mass gap in ordinary, pure Yang-Mills theory. For this we shall investigate how the results derived in the supersymmetric context can be extended to ordinary Yang-Mills theory. In particular, we inquire whether a proper variant of the N=2 structure can be identified, and whether a gauge invariant and renormalizable mass scale can be explicitly introduced.

Since the number of degrees of freedom in ordinary Yang-Mills theory is insufficient for identifying structures such as the N=2 supersymmetry, we need to introduce additional variables. For this we investigate the nonperturbative structure of Yang-Mills path integral in the background field formalism [3]. We first separate the classical background fields from their quantum fluctuations, and explicitly sum the path integral over these classical backgrounds. We then represent this summation as a path integral over a set of auxiliary variables. This promotes the classical background configurations into *off-shell* degrees of freedom, and introduces a large number of new variables. These variables determine a topological quantum field theory that describes the space of classical solutions of the original Yang-Mills theory. We apply standard BRST arguments to extend this topological field theory so that it involves a number of parameters, with

BRST symmetry ensuring that the theory is independent of these parameters. For a particular value of these parameters we recover explicitly the original summation over classical Yang-Mills fields in the background field formalism. But for other values of these parameters our effective field theory admits other interpretations.

We shall be interested in two a priori different sets of classical field configurations, the (anti)selfdual ones and those that describe all possible solutions to the classical Yang-Mills equations.

We first consider the (anti)selfdual configurations. For a finite Euclidean action these are the instantons, and the pertinent topological field theory is the one introduced by Witten to describe Donaldson's invariants [4]. When we sum our Yang-Mills path integral over all (anti)selfdual configurations in the background field formalism, we then obtain an effective field theory that describes a coupling between the original Yang-Mills quantum degrees of freedom and Witten's topological Yang-Mills theory. Furthermore, when we employ unitarity arguments to introduce a twist in the Lorentz transformations we can identify our topological sector as the minimal N=2 supersymmetric Yang-Mills theory.

This appearance of a N=2 structure in ordinary Yang-Mills theory is certainly most interesting. We view it as an indication that the results derived in [1] could be extended to explain color confinement in ordinary, pure Yang-Mills theory. However, we first need to understand how a mass gap appears. For this we observe that besides the N=2 supersymmetry, the topological sector of our theory also admits an additional SU(N) gauge invariance implying that our effective theory actually has an extended SU(N)×SU(N) gauge invariance. Both of these additional symmetries may then be explicitly broken to the original SU(N) symmetry of the ordinary Yang-Mills theory. This opens the possibility that we can introduce additional terms to our effective action, provided these additional terms are consistent with the SU(N) gauge invariance of our original Yang-Mills theory. Such additional terms might then have interesting physical consequences to the original Yang-Mills theory.

One possibility to break part of these extra symmetries is, that following [1] we add an explicit mass term to the scalar superfield. This breaks the N=2 supersymmetry in the topological sector explicitly into a N=1 supersymmetry. Since this mass term is renormalizable, in the present context it might also provide an alternative to the

standard Higgs mechanism for constructing a renormalizable theory of massive gauge vectors. However, a disadvantage of such a scalar superfield mass term is that it does not break the SU(N)×SU(N) gauge invariance into the physical SU(N). For this reason we propose an alternative, which arises when we add a mass term to the field that describes quantum fluctuations around the original classical background. This mass term explicitly breaks both the N=2 supersymmetry and the SU(N)×SU(N) gauge symmetry into the physical SU(N) gauge symmetry. We argue that this mass term is also consistent with unitarity and power counting renormalizability, suggesting that it might have some relevance to the expected generation of a mass gap in ordinary quantum Yang-Mills theory. Furthermore, this mass term might also provide an alternative to the Higgs mechanism for constructing massive gauge vectors.

We then proceed to investigate the effective theory that appears when we sum over all possible solutions to the Yang-Mills equation. In addition of instantons, now the pertinent topological quantum field theory also describes classical solutions that do not necessarily satisfy the first-order (anti)selfduality equation. As a consequence, instead of the N=2 supersymmetry we now find the Parisi-Sourlas supersymmetry [5] in the topological sector. Furthermore, as in the (anti)selfdual case we find an additional, independent SU(N) gauge invariance in this topological sector, and again both of these additional symmetries can be broken explicitly into the physical SU(N) gauge symmetry. For this, we again introduce a mass term to the field that describes quantum fluctuations around the classical Yang-Mills background.

The emergence of a Parisi-Sourlas supersymmetry is particularly interesting, since it suggests that ordinary Yang-Mills theory admits a subsector which exhibits the D=4 → D=2 Parisi-Sourlas dimensional reduction [5]. Indeed, previously it has been conjectured that such a subsector should exist [6] and it should dominate the Yang-Mills theory in the infrared limit and explain color confinement: When planar Wilson loops are restricted to this subsector they exhibit an area law as a consequence of the D=4 → D=2 dimensional reduction [7], [8]. The present construction reveals that this structure actually appears in ordinary Yang-Mills theory. We verify that at large spatial distances the propagator for the Parisi-Sourlas gauge field behaves like $\mathcal{O}(p^{-4})$ consistent with a linearly growing potential, and provide arguments which support that this behaviour indeed dominates in the infrared limit. We also explain how the ensuing string tension emerges from a

cohomological construction.

Finally, we confirm that the summation over (anti)selfdual configurations provides a good approximation to the summation over all possible classical solutions. This suggests that the approach to confinement in supersymmetric theories developed in [1] can also be adopted to explain color confinement in ordinary Yang-Mills theory. In particular, our results suggest that the N=2 supersymmetry approach to confinement is intimately connected to the picture of color confinement due to randomly distributed color-electric and color-magnetic fields with the ensuing Parisi-Sourlas dimensional reduction as developed in [6], [7], [8].

In the next section we shall shortly review the background field formalism. We also describe how it can be applied to implement a summation over classical field configurations. In sections 3. and 4. we consider the approximation that arises when we sum over all (anti)selfdual connections. In section 5. we explain how a gauge invariant mass term can be introduced, and argue that this mass term also preserves power-counting renormalizability. In section 6. we investigate unitarity of our theory. We find a manifestly unitary representation by re-interpreting the action of Lorentz group on our fields, and conclude that ordinary Yang-Mills theory actually *contains* the N=2 supersymmetric Yang-Mills theory. In sections 7. and 8. we proceed to the general case, obtained when we sum over all possible classical solutions. Now we find an effective action with Parisi-Sourlas supersymmetry. This suggests that ordinary Yang-Mills theory admits a nonperturbative sector that exhibits the D=4 → D=2 Parisi-Sourlas dimensional reduction. We argue that this sector dominates in the infrared limit, indicating that color confinement could be a consequence of an effective dimensional reduction. Finally, in section 9. we verify that the results obtained in the (anti)selfdual approximation are consistent with those obtained when we sum over all possible solutions. This means that the N=2 picture indeed provides a good desription of the exact theory. It suggests that the recent approach to confinement in the N=2 supersymmetric theory may be directly relevant for describing color confinement also in ordinary Yang-Mills theory.

2 Background field quantization

We shall consider a $SU(N)$ Yang-Mills theory with gauge field C_μ^a and field strength

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu C_\nu - \partial_\nu C_\mu + [C_\mu, C_\nu]$$

We normalize the Lie algebra generators T^a so that the classical action is

$$S_{YM}(C) = \int \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \equiv \int \frac{1}{4} F_{\mu\nu}^2 \tag{1}$$

We shall be interested in the (Euclidean) partition function

$$Z = \int [dC] \exp\{-S_{YM}(C)\} \tag{2}$$

in the background field formalism. For this we represent the gauge field C_μ as a linear combination

$$C_\mu = A_\mu + Q_\mu \tag{3}$$

where we select A_μ to satisfy the classical equation of motion,

$$D_\mu F_{\mu\nu}(A) = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 \tag{4}$$

The field Q_μ then describes perturbative quantum fluctuations around the classical field A_μ. This means that the classical field A_μ satisfies nontrivial boundary conditions, while the boundary conditions for the fluctuation field Q_μ are trivial. For example in the case of an instanton the second Chern class

$$\mathrm{Ch}_2(A) = \int F_{\mu\nu} \tilde{F}_{\mu\nu}(A)$$

for A_μ is nontrivial and coincides with the second Chern class for C_μ.

If we define

$$G_{\mu\nu} = D_\mu Q_\nu - D_\nu Q_\mu \tag{5}$$

where the covariant derivative is *w.r.t.* the classical field A_μ, we can write the action (1) as

$$
\begin{aligned}
-S_{YM}(A+Q) = {} & -\frac{1}{4} F_{\mu\nu}^2 - Q_\mu D_\nu F_{\mu\nu} - \frac{1}{2} F_{\mu\nu}[Q_\mu, Q_\nu] \\
& -\frac{1}{4} G_{\mu\nu}^2 - \frac{1}{2} G_{\mu\nu}[Q_\mu, Q_\nu] - \frac{1}{4} [Q_\mu, Q_\nu]^2
\end{aligned}
\tag{6}
$$

Since A_μ solves the Yang-Mills equation (4), the Q-linear term $Q_\mu D_\nu F_{\mu\nu}$ in (6) actually vanishes. However, eventually we shall promote A_μ to an off-shell field so that it will cease to be constrained by (4). In that case the Q-linear term does not vanish, and in anticipation of this we have included it also here.

Notice that there is certain latitude in defining the gauge transformations of the fields that appear on the *r.h.s.* of (3). Here we have selected the gauge transformation of the classical field A_μ to coincide with the standard gauge transformation of a $SU(N)$ gauge field

$$A_\mu \;\to\; UA_\mu U^{-1} \;+\; U\partial_\mu U^{-1} \tag{7}$$

so that the fluctuation field Q_μ gauge transforms homogeneously, like a Higgs field

$$Q_\mu \;\to\; UQ_\mu U^{-1} \tag{8}$$

This implies that each term in (6) is separately gauge invariant.

In terms of these background variables the path integral (2) becomes

$$Z_{YM} \;=\; \sum_{A_\mu} \int [dQ]\exp\{\int -\frac{1}{4}F_{\mu\nu}^2 \;-\; Q_\mu D_\nu F_{\mu\nu} \;-\; \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu]$$

$$-\; \frac{1}{4}G_{\mu\nu}^2 \;-\; \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] \;-\frac{1}{4}\,[Q_\mu, Q_\nu]^2\,\} \tag{9}$$

The summation extends over all solutions A_μ of the Yang-Mills equation of motion, and we remind that the integral over Q_μ is subject to trivial boundary conditions. Notice that since the moduli space of the classical field A_μ is generically nontrivial (*e.g.* for a k-instanton it is a $8k-3$ dimensional manifold), the summation over A_μ should actually be viewed as an integration over the relevant moduli.

Heuristically, we can represent the summation over A_μ as

$$\sum_{A_\mu} \;=\; \int [dA]\delta(D_\mu F_{\mu\nu})\left|\,\det ||\frac{\delta D_\mu F_{\mu\nu}}{\delta A_\rho}||\,\right| \;\approx\; \sum_{A_\mu}\left|\,sign\det ||\frac{\delta DF}{\delta A}||\,\right| \tag{10}$$

but unfortunately, due to the absolute values it is usually quite difficult to implement (10) in (9). For this reason, we use the common practice and replace the summation over A_μ by the more manageable

$$\sum_{A_\mu} \;\to\; \int [dA]\delta(D_\mu F_{\mu\nu})\det ||\frac{\delta D_\mu F_{\mu\nu}}{\delta A_\rho}|| \;\approx\; \sum_{A_\mu} sign\det ||\frac{\delta DF}{\delta A}|| \tag{11}$$

and *approximate* (9) by

$$Z_{YM} \approx \int [dA][dQ] \delta(D_\mu F_{\mu\nu}) \det ||\frac{\delta D_\mu F_{\mu\nu}}{\delta A_\rho}|| \exp\{\int -\frac{1}{4}F_{\mu\nu}^2 - Q_\mu D_\nu F_{\mu\nu} - \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu]$$

$$- \frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] - \frac{1}{4}[Q_\mu, Q_\nu]^2 \} \tag{12}$$

$$\approx \sum_{A_\mu} sign \det ||\frac{\delta DF}{\delta A}|| \left\{ \int [dQ] \exp\{-\int \frac{1}{4}F_{\mu\nu}^2(A+Q)\} \right\} \tag{13}$$

In order to interpret (12), (13) mathematically, we resort to a finite dimensional analogue and view the Yang-Mills action $S_{YM}(A+Q)$ in (13) as an infinite dimensional counterpart of a (nondegenerate) Morse function $H(x)$. A sum such as (10) counts the number of its critical points, and is bounded from below by the sum of Betti numbers B_n of the underlying manifold (in our case the gauge orbit space $\mathcal{A}/\mathcal{G}$)

$$\sum_{dH=0} 1 \geq \sum_n B_n$$

On the other hand, a sum such as the *r.h.s.* of (11) is independent of the Morse function and according to the Poincaré-Hopf theorem [9] it coincides with the Euler character $\mathcal{X}$ of the manifold

$$\sum_{dH=0} sign \det ||\frac{\partial^2 H}{\partial x_a \partial x_b}|| = \sum_n (-)^n B_n \equiv \mathcal{X} \tag{14}$$

If H is a perfect Morse function these two quantities coincide, but for a general Morse function they are different since in general the fluctuation matrix $\partial_{ab}H$ admits an odd number of zeromodes for some of the critical points x_a.

For a compact finite dimensional Riemannian manifold the Euler character (14) coincides with the partition function of the de Rham supersymmetric quantum mechanics [10], [11]. This partition function can be evaluated exactly, *e.g.* by localizing the corresponding path integral to the Euler class of the manifold. In this way we obtain the standard relation between the Poincaré-Hopf and Gauss-Bonnet-Chern theorems.

On the other hand, the summation that appears in (12) is an infinite dimensional generalization of a sum of the form

$$\sum_{dH=0} sign \det ||\frac{\partial^2 H}{\partial x_a \partial x_b}|| \exp\{-T\mathcal{H}\}$$

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where $\mathcal{H}$ corresponds to the Q-integral in (13). When H and $\mathcal{H}$ coincide, we obtain a quantity that appears in an equivariant version of the Poincaré-Hopf theorem [12]. There is also an equivariant version of the Gauss-Bonnet-Chern theorem, and as in conventional Morse theory one can derive a relation between these two theorems using an equivariant version of the de Rham supersymmetric quantum mechanics [12]. The pertinent path integral is intimately related to a standard Hamiltonian path integral, with the Morse function H interpreted as the Hamiltonian function. Indeed, this interrelationship between equivariant Morse theory and standard Hamiltonian path integrals can be utilized to evaluate certain Hamiltonian partition functions exactly, using localization techniques [13]. This leads to the Duistermaat-Heckman integration [14] formula and its quantum mechanical generalizations [12], [13].

The present mathematical interpretations suggest that (12) may in fact provide a relatively good approximation of the original partition function (9). Proceeding with (12) we at least obtain a reliable lower-bound estimate of the partition function. In the following we shall investigate (12), using the insight provided by these mathematical analogues.

The Yang-Mills equation (4) admits various different kinds of solutions, and the most interesting ones are the instantons. For an instanton the fluctuation matrix in (10), (12) admits only non-negative eigenvalues. Consequently in an instanton approximation the absolute signs in (10) are not relevant, implying that (12) describes adequately the pertinent contribution to the original path integral (provided we properly account for the zero modes in the fluctuation matrix). This is intimately related to the fact, that for instantons the Yang-Mills equation simplifies to the first-order (anti)selfduality equations

$$F^{\pm} \; = \; F_{\mu\nu} \; \pm \; \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \; = \; 0 \tag{15}$$

where the $+$ refers to anti-selfdual configurations and $-$ to selfdual ones.

Formal Morse theory arguments suggest that the Poincaré-Hopf representation of the Euler character $\mathcal{X}$ on $\mathcal{A}/\mathcal{G}$ should be independent of the Morse function. This means, that we should be able to utilize the (anti)selfduality equations (15) to introduce the following alternative realizations of the Euler character,

$$\mathcal{X}(\mathcal{A}/\mathcal{G}) \; = \; \sum_{DF=0} sign \det ||\frac{\delta DF}{\delta A}|| \; = \; \sum_{F^-=0} sign \det ||\frac{\delta F^-}{\delta A}|| \; = \; \sum_{F^+=0} sign \det ||\frac{\delta F^+}{\delta A}||$$

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$$\approx \int [dA]\delta(F^{\pm}) \det ||\frac{\delta F^{\pm}}{\delta A}|| \tag{16}$$

Furthermore, since the (anti)selfdual configurations solve the original Yang-Mills equation, this suggests that we might also obtain a reliable approximation to (9) if instead of (12), (13) we use

$$Z_{YM} \approx \sum_{F^{\pm}=0} sign \det ||\frac{\delta F^{\pm}}{\delta A}|| \left\{ \int [dQ] \exp\{ \int -\frac{1}{4}F^2_{\mu\nu}(A+Q)\} \right\}$$

$$\approx \int [dA][dQ]\delta(F^{\pm}) \det ||\frac{\delta F^{\pm}}{\delta A}|| \exp\{ \int -\frac{1}{4}F^2_{\mu\nu} - Q_\mu D_\nu F_{\mu\nu} - \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu]$$

$$- \frac{1}{4}G^2_{\mu\nu} - \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] - \frac{1}{4}[Q_\mu, Q_\nu]^2 \} \tag{17}$$

where the zeromodes can again be accounted for *e.g.* by introducing collective coordinates.

Notice that since the (anti)selfdual fluctuation matrices that appear in (17) are first-order operators, these matrices admit an infinite number of negative eigenvalues. In particular this means that (17) is *a priori* quite different from the usual semiclassical approximation, where one evaluates the fluctuations around *both* instantons and anti-instantons using the path integral (12). There, the fluctuation matrix is the original Yang-Mills one and for (anti)selfdual configurations it is positive semidefinite.

Finally, we inquire how (12) should be corrected so that instead of approximating the original path integral (9), we get exact results. For this, we need to account for the absolute signs in the fluctuation determinants.

Formally, the sign of the fluctuation determinant in (10) is

$$sign \det ||\frac{\delta DF}{\delta A}|| = \exp\{i\pi \sum_{\lambda_n<0} 1\} \tag{18}$$

where λ_n are eigenvalues of the fluctuation matrix,

$$\frac{\delta DF}{\delta A}\psi_n = \lambda_n \psi_n$$

We write

$$\sum_{\lambda_n<0} 1 = \frac{1}{2}\sum_{\lambda_n} 1 - \frac{1}{2}\sum_{\lambda_n} sign(\lambda_n)$$

$$= \frac{1}{2}\zeta_{YM} - \frac{1}{2}\eta_{YM}$$

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where ζ_{YM} is the ζ-function of the fluctuation matrix

$$\zeta_{YM}(s) = \sum_{\lambda_n} |\lambda_n|^{-s}$$

and η_{YM} is its η-invariant,

$$\eta_{YM}(s) = \sum_{\lambda_n} sign(\lambda_n)|\lambda_n|^{-s}$$

both evaluated at $s = 0$ and at the background configuration A_μ. Consequently the phase is

$$sign \det ||\frac{\delta DF}{\delta A}|| = \exp\{\frac{i}{2}\pi(\zeta_{YM} - \eta_{YM})\} \tag{19}$$

This implies that we obtain (9) if we improve (12) to

$$Z_{YM} = \int [dA][dQ]\delta(DF) \det ||\frac{\delta DF}{\delta A}|| \exp\{\int -\frac{1}{4}F_{\mu\nu}^2 - Q_\mu D_\nu F_{\mu\nu} - \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu]$$

$$- \frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] - \frac{1}{4}[Q_\mu, Q_\nu]^2 - \frac{i}{2}\pi(\zeta_{YM} - \eta_{YM})(A)\} \tag{20}$$

For the instanton approximation (17) we introduce similarly

$$Z_{YM} \approx \int [dA][dQ]\delta(F^\pm) \det ||\frac{\delta F^\pm}{\delta A}|| \exp\{\int -\frac{1}{4}F_{\mu\nu}^2 - Q_\mu D_\nu F_{\mu\nu} - \frac{1}{2}F_{\mu\nu}[Q_\mu, Q_\nu]$$

$$- \frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}G_{\mu\nu}[Q_\mu, Q_\nu] - \frac{1}{4}[Q_\mu, Q_\nu]^2 - \frac{i}{2}\pi(\zeta_{YM}^\pm - \eta_{YM}^\pm)(A)\} \tag{21}$$

where $\zeta_{YM}^\pm$ and $\eta_{YM}^\pm$ now denote the ζ-function and the η-invariant of the (anti)selfdual fluctuation matrix. However, now we do not necessarily expect to obtain an improvement of (17). In fact, since the fluctuation determinant in (12) is non-negative for (anti)selfdual configurations, it appears that when (21) is summed over both selfdual and anti-selfdual configurations we reproduce the standard instanton approximation of (12).

Since the fluctuation matrix of the original Yang-Mills equation is an elliptic second-order operator, the number of its negative eigenvalues is restricted and consequently (18) is quite sufficient. However, since the fluctuation matrices of the (anti)selfdual equations are first order operators, the regulated expressions are more appropriate.

In general, the evaluation of the phase factors ζ_{YM} and η_{YM} is prohibitively complicated and consequently we shall proceed with the more manageable (12) and (17). But

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since the contribution from these phase factors is additive, we conjecture that the qualitative aspects of our analysis remain largely intact. Furthermore, in the following we shall be mostly interested in the infrared limit of the Yang-Mills theory. In this limit we expect both ζ_{YM} and η_{YM} to become irrelevant operators, and we conjecture that these phase factors are relevant only when we attempt to compute higher order corrections.

3 Topological Yang-Mills theory

We shall first investigate the (anti)selfdual approximation (17). We wish to promote the classical field A_μ into an off-shell field, by representing the δ-function and the determinant as a path integral over a set of auxiliary variables. For this, we first need an appropriate representation of the Euler character (16). For definiteness we specialize to the anti-selfdual configurations $F^+ = 0$.

The path integral representation of the Euler character

$$\mathcal{X}(\mathcal{A}/\mathcal{G}) \;=\; \sum_{F^+=0} sign \det ||\frac{\delta F^+}{\delta A}|| \tag{22}$$

has been investigated by Atiyah and Jeffrey [15]. They showed, that (22) coincides with the partition function of topological Yang-Mills theory [4] in the Mathai-Quillen formalism [14]. Their construction can be viewed as a direct generalization of the familiar result that the Euler character of a compact Riemannian manifold can be represented by the partition function of the de Rham supersymmetric quantum mechanics [10], [11].

The partition function of topological Yang-Mills theory is an example of a cohomological path integral of the form

$$Z_{TYM} \;=\; \int \exp\left(\{\Omega_0, \Psi\}\right) \tag{23}$$

where Ω_0 is a nilpotent BRST operator and Ψ is a gauge fermion. Standard arguments imply that this path integral describes only the cohomology classes of Ω_0, and it is formally invariant under local variations of Ψ.

Atiyah and Jeffrey [15] constructed a one-parameter family of Ψ's that interpolates between the Gauss-Bonnet-Chern and the Poincaré-Hopf representatives of the Euler class on $\mathcal{A}/\mathcal{G}$. In particular, for a definite value of the parameter their action reproduces

the original action of topological Yang-Mills theory [4]. For this, we use the notation of Ouvry, Stora and van Baal [16] (except that we denote by **u** and **v** the standard ghosts for gauge fixing) and introduce a graded symplectic manifold with the following canonical variables

form degree	EVEN (q,p)	ODD (q,p)
0-form	φ , π	**u, v**
1-form	A, E	ψ, $\mathcal{X}$
2-form	b, c	$\bar{\psi}$, $\bar{\mathcal{X}}$
4-form	$\bar{\varphi}$, $\bar{\pi}$	β, γ

The graded Poisson brackets of these variables are

$$\{p^a, q^b\} = -\delta^{ab} \tag{24}$$
$$\{p^a_\mu, q^b_\nu\} = -\delta^{ab}_{\mu\nu}$$
$$\{p^a_{\mu\nu}, q^b_{\rho\sigma}\} = -\frac{1}{4}\delta^{ab}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma})$$

where the 2-form bracket explicitly accounts for the antisymmetry and selfduality of the corresponding variables.

The nilpotent BRST operator Ω_0 that computes the cohomology of the topological Yang-Mills theory can be represented as a linear combination [16], [8],

$$\Omega_0 = \Omega_{TOP} + \Omega_{YM} \tag{25}$$

Here

$$\Omega_{YM} = \mathbf{u}\mathcal{G} + \frac{1}{2}\mathbf{v}[\mathbf{u}, \mathbf{u}] + h\bar{\mathbf{u}} \tag{26}$$

is the standard nilpotent BRST operator that fixes the SU(N) gauge invariance, with

$$\mathcal{G} = D_\mu E_\mu + [\varphi, \pi] + [\bar{\varphi}, \bar{\pi}] + [\beta, \gamma] + [\psi_\mu, \mathcal{X}_\mu] + [\bar{\psi}_{\mu\nu}, \bar{\mathcal{X}}_{\mu\nu}] + [b_{\mu\nu}, c_{\mu\nu}]$$

the Gauss law operator which generates the gauge transformations of the various fields,

$$[\mathcal{G}^a, \mathcal{G}^b] = f^{abc}\mathcal{G}^c$$

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The other operator

$$\Omega_{TOP} = \psi_\mu E_\mu + \varphi(D_\mu \mathcal{X}_\mu + [\bar{\psi}_{\mu\nu}, c_{\mu\nu}] + \mathbf{v}) + b_{\mu\nu}\bar{\mathcal{X}}_{\mu\nu} + \gamma[\varphi, \bar{\varphi}] + \beta\bar{\pi} \qquad (27)$$

is the BRST operator for the topological symmetry. It is equivariantly nilpotent,

$$\Omega^2_{TOP} = -2\varphi\mathcal{G}$$

Since this generates a gauge transformation with φ as the gauge parameter, Ω_{TOP} is then nilpotent on the gauge orbit $\mathcal{A}/\mathcal{G}$.

From [17], [8] we conclude, that the *complete* BRST operator of the topological Yang-Mills theory is the linear combination of (25) with another nilpotent operator Ω_{gf} which is necessary for gauge fixing the various symmetries,

$$\Omega_{full} = \Omega_0 + \Omega_{gf} \qquad (28)$$

In (26) we have included the pertinent Ω_{gf} that fixes the standard SU(N) gauge symmetry, it corresponds to the $h\bar{\mathbf{u}}$ term. In the general case the operators Ω_0 and Ω_{gf} also anticommute, ensuring the nilpotency of Ω_{full}. The structure of Ω_{gf} in the topological sector has been explained *e.g.* in [8], and will be discussed in section 9 of the present paper. Like the corresponding term in (26) it is trivial, and for the present purposes it is sufficient to consider only Ω_0. Consequently in the following we shall not consider Ω_{gf} explicitly in the topological sector, it is enough for us to assume that the topological gauge invariances are fixed by some appropriate gauge condition.

We construct the action of topological Yang-Mills theory in the path integral (23), by first introducing four different gauge fermions Ψ_i

$$\begin{aligned} \Psi_1 &= \bar{\psi} \wedge b \\ \Psi_2 &= \bar{\psi} \wedge F^+ \\ \Psi_3 &= \star\bar{\varphi} \wedge D \star \psi \\ \Psi_4 &= \beta \wedge [\varphi, \star\bar{\varphi}] \end{aligned} \qquad (29)$$

Here $\star$ denotes the Hodge duality transformation. We introduce four numerical parameters α_i and define the gauge invariant topological action

$$S_{TOP} = \sum_{i=1}^{4} \alpha_i \{\Omega_{TOP}, \Psi_i\} \qquad (30)$$

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By substituting (29) and eliminating the auxiliary field b by a Gaussian integration in (23), we obtain

$$-S_{TOP} = \alpha_1 \varphi[\bar\psi, \bar\psi] + \alpha_2 \left[-\frac{\alpha_2}{\alpha_1} \frac{1}{4} F^+ \wedge F^+ + D\psi^+ \wedge \bar\psi \right]$$

$$+ \alpha_3 \left(\star\beta \wedge D \star \psi + \star\bar\varphi \wedge [\psi, \star\psi] - \star\bar\varphi D \star D\varphi \right) + \alpha_4 \left([\varphi, \star\bar\varphi]^2 + \varphi[\star\beta, \star\beta] \right) \star 1 \quad (31)$$

Different values of α_i label different representations of the theory, and by general arguments the ensuing path integral (23) should be independent of these parameters. For example, if we select

$$\alpha_1 = -\alpha_2 = \alpha_3 = 1, \qquad \alpha_4 = 0$$

we find the action presented in [16]. On the other hand, the action originally introduced by Witten [4] emerges if we select

$$\alpha_1 = 4, \qquad \alpha_2 = -4, \qquad \alpha_3 = 1, \qquad \alpha_4 = \frac{1}{2} \quad (32)$$

and use the following identification of variables between the notations in [16] and [4],

OSvB	Witten
A	A
ψ	$i\psi$
φ	$i\phi$
$\bar\psi$	$\frac{1}{4}\mathcal{X}$
$\star\bar\varphi$	$-\frac{i}{2}\lambda$
$\star\beta$	η

The result is

$$-S_W = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} F_{\mu\nu} \tilde{F}_{\mu\nu} - \frac{1}{2} \phi D_\mu^2 \lambda + i\eta D_\mu \psi_\mu - i D_\mu \psi_\nu \mathcal{X}_{\mu\nu}$$

$$+ \frac{i}{8} \phi[\mathcal{X}_{\mu\nu}, \mathcal{X}_{\mu\nu}] + \frac{i}{2} \lambda[\psi_\mu, \psi_\mu] + \frac{i}{2} \phi[\eta, \eta] + \frac{1}{8}[\phi, \lambda]^2 \quad (33)$$

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In [15] Atiyah and Jeffrey showed, that the corresponding path integral (23) yields the Euler character (22) on $\mathcal{A}/\mathcal{G}$. For this, we return to the notation of [16] and select

$$\alpha_2 = -1 , \qquad \alpha_4 = 0$$

but leave α_1 and α_3 arbitrary. We again eliminate b by a Gaussian integration and find for the path integral (23)

$$Z_{TYM} = \int [dA] \dots [d\lambda][\sqrt{\frac{1}{4\pi\alpha_1}}] \exp\{- \int \frac{1}{4\alpha_1}(F^+)^2 + D\psi^+ \wedge \bar\psi - \alpha_1\varphi[\bar\psi, \bar\psi]$$

$$- \alpha_3(\star\beta \wedge D \star \psi + \star\bar\varphi \wedge [\psi, \star\psi] - \star\bar\varphi D \star D\varphi)\} \tag{34}$$

The parameter α_3 can be eliminated by redefining β and $\bar\varphi$, and we get

$$Z_{TYM} = \int [dA][d\psi][d\bar\psi][d\varphi][\sqrt{\frac{1}{4\pi\alpha_1}}] \exp\{- \int \frac{1}{4\alpha_1}(F^+)^2 + D\psi^+ \wedge \bar\psi$$

$$- \alpha_1 \left(\frac{1}{\star D \star D} \star [\psi, \star\psi] \right) \cdot [\bar\psi, \bar\psi] \} \tag{35}$$

Here

$$R = \frac{1}{\star D \star D} \star [\psi, \star\psi] \tag{36}$$

can be identified as the curvature two-form on $\mathcal{A}/\mathcal{G}$ when we restrict ψ to be horizontal on the gauge orbit space,

$$D \star \psi = 0 \tag{37}$$

which follows as a δ-function constraint when we integrate over β in (34). (We assume that a proper gauge fixing - determined by Ω_{gf} in (28) - has been introduced, so that the integral in (35) extends only over the gauge orbit $\mathcal{A}/\mathcal{G}$.) Indeed, if we introduce the connection

$$\Gamma = \frac{1}{\star D \star D} \star D \star \psi \tag{38}$$

and define the exterior derivative by

$$d = \psi_\mu^a \frac{\delta}{\delta A_\mu^a}$$

we find that the curvature two-form

$$R = d\Gamma + \Gamma \wedge \Gamma$$

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coincides with (36) when we restrict to the horizontal bundle (37).

According to general arguments, the path integral (35) is at least formally independent of α_1 and we can interpret it by considering various limits:

When $\alpha_1 \to \infty$ we find that (35) reduces to the Gauss-Bonnet-Chern representation of the Euler character on $\mathcal{A}/\mathcal{G}$,

$$Z_{TYM} = \int [dA][d\psi] \mathrm{Pf}(R) \tag{39}$$

where R is the curvature two-form (36).

On the other hand, when $\alpha_1 \to 0$ we get

$$Z_{TYM} = \int [dA] \delta(F^+) \det ||D^+|| \approx \sum_{F^+=0} sign \det ||\frac{\delta F^+}{\delta A}|| \tag{40}$$

which is the Poincaré-Hopf representation (22) of the Euler character on $\mathcal{A}/\mathcal{G}$. As a consequence we have a generalization of the finite dimensional relation between the Gauss-Bonnet-Chern and Poincaré-Hopf theorems, and in particular (36) is indeed the curvature two-form on $\mathcal{A}/\mathcal{G}$.

4   The Instanton Approximation

We shall now apply the results of the previous section to investigate the anti-selfdual approximation (17) to the Yang-Mills partition function (2),

$$Z_{YM} = \sum_{F^+=0} sign \det ||\frac{\delta F^+}{\delta A}|| \left\{ \int [dQ] \exp\{\int -\frac{1}{4} F_{\mu\nu}^2 (A+Q)\} \right\} \tag{41}$$

Evidently the selfdual approximation is similar, and there is no need to consider it explicitly.

For this we define the following more general path integral

$$Z_{YM}(\alpha_1, \alpha_3) = \int [dQ][dA]...[d\beta] \exp\{-S_{YM}(A+Q) - S_{AJ}(A; \alpha_1, \alpha_3)\} \tag{42}$$

where S_{YM} is the $A_\mu + Q_\mu$ dependent Yang-Mills background field action that appears in (41) and S_{AJ} is the A_μ dependent Atiyah-Jeffrey representation of the topological Yang-Mills action that appears in (34). Since the $\alpha_1 \to 0$ limit of (35) localizes to the

Poincaré-Hopf representation (40) of the Euler character, we conclude that we obtain (41) when $\alpha_1 \to 0$ in (42),

$$\lim_{\alpha_1 \to 0} Z_{YM}(\alpha_1, \alpha_3) = Z_{YM}$$

A priori, the path integral (42) depends nontrivially on the parameters α_1 and α_3. However, we shall now argue that (42) is actually *independent* of α_1 and α_3 so that it coincides with (41) independently of these parameters. More generally, we shall argue that independently of the parameters α_i we can write (41) as

$$Z_{YM} = \int [dQ][dA]...[d\beta] \exp\left(\int -\frac{1}{4} F_{\mu\nu}^2(A+Q) - \sum_{i=1}^{4} \alpha_i \{\Omega_{TOP}, \Psi_i\} \right) \qquad (43)$$

where Ω_{TOP} is the topological BRST operator (27) and Ψ_i are the gauge fermions defined in (29).

Notice in particular, that the second term in (43) depends only on the classical field A_μ, and coincides with the action (30) of topological Yang-Mills theory.

In order to demonstrate the α_i independence of (43) we first introduce a new variable P_μ^a which is conjugate to the fluctuation field Q_μ^a,

$$\{P_\mu^a, Q_\nu^b\} = -\delta_{\mu\nu}^{ab}$$

We then define the following linear combinations,

$$
\begin{aligned}
A_\mu^+ &= A_\mu + Q_\mu \\
A_\mu^- &= A_\mu \\
E_\mu^+ &= P_\mu \\
E_\mu^- &= E_\mu - P_\mu
\end{aligned}
\qquad (44)
$$

Since the only nonvanishing Poisson brackets are

$$\{E_\mu^\pm, A_\nu^\pm\} = -\delta_{\mu\nu} \qquad (45)$$

we conclude that A_μ^+ and A_μ^- are actually two *independent* gauge fields.

More generally, we extend these $\pm$ variables into a one-parameter family of $\pm$ variables by introducing a canonical conjugation generated by

$$\Phi = -\tau E_\mu Q_\mu$$

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where τ is a parameter. This gives

$$
\begin{aligned}
A_\mu^+ &\rightarrow e^{-\Phi} A_\mu^+ e^{\Phi} &=& \quad A_\mu + (1 - \tau)Q_\mu \\
A_\mu^- &\rightarrow e^{-\Phi} A_\mu^- e^{\Phi} &=& \quad A_\mu - \tau Q_\mu \\
E_\mu^+ &\rightarrow e^{-\Phi} E_\mu^+ e^{\Phi} &=& \quad P_\mu + \tau E_\mu \\
E_\mu^- &\rightarrow e^{-\Phi} E_\mu^- e^{\Phi} &=& \quad -P_\mu + (1 - \tau)E_\mu
\end{aligned}
\tag{46}
$$

and reproduces (44) for $\tau = 0$. Since this is a canonical transformation, the Poisson brackets (45) are preserved. Notice in particular, that independently of τ both fields $A_\mu^\pm$ gauge transform like a SU(N) gauge field,

$$
A_\mu^\pm = U A_\mu^\pm U^{-1} + U \partial_\mu U^{-1}
$$

The present construction implies that the action in (43) separates into two independent contributions. The first term depends only on the gauge field A_μ^+ and the second term depends only on the gauge field A_μ^-,

$$
S_{YM}(A^+) + S_{TOP}(A^-) = \frac{1}{4}F_{\mu\nu}^2(A^+) + \sum_{i=1}^{4} \alpha_i \{\Omega_{TOP}^-, \Psi_i\}(A^-)
\tag{47}
$$

where the BRST operator Ω_{TOP}^- now acts only on the fields A^- and E^-,

$$
\Omega_{TOP}^- = \psi_\mu E_\mu^- + \varphi(D_\mu^- \mathcal{X}_\mu + [\bar{\psi}_{\mu\nu}, c_{\mu\nu}] + \rho) + b_{\mu\nu}\bar{\mathcal{X}}_{\mu\nu} + \gamma[\varphi, \bar{\varphi}] + \beta\bar{\pi}
\tag{48}
$$

Here D_μ^- is covariant derivative w.r.t. the gauge field A_μ^-. For $\tau = 0$ (48) reduces to (25) except for the (irrelevant) shift $E_\mu \rightarrow E_\mu - P_\mu$, and since τ only parametrizes a canonical transformation this establishes the asserted α_i-independence of the path integral (43). In particular, by selecting $\alpha_2 = -1$, $\alpha_4 = 0$ and taking the $\alpha_1 \rightarrow 0$ limit we obtain our anti-selfdual approximation (41).

Notice that the present construction also implies, that instead of the original SU(N) gauge symmetry we now have two *independent* SU(N) gauge symmetries acting on the fields $A^\pm$ respectively, and we have a SU(N)×SU(N) gauge theory. However, the gauge fields $A_\mu^\pm$ are not entirely independent. Since

$$
A_\mu^+ - A_\mu^- = Q_\mu
\tag{49}
$$

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obeys trivial boundary conditions the gauge fields $A_\mu^\pm$ are subject to the condition that their second Chern classes coincide,

$$\int F_{\mu\nu}\tilde{F}_{\mu\nu}(A^+) = \int F_{\mu\nu}\tilde{F}_{\mu\nu}(A^-) \tag{50}$$

This also ensures that the path integral in our anti-selfdual approximation,

$$Z_{YM} = \int [dA^+][dA^-]...[d\beta]\exp\{\int -\frac{1}{4}F_{\mu\nu}^2(A^+) - \sum_{i=1}^4 \alpha_i\{\Omega_{TOP}^-, \Psi_i\}(A^-)\} \tag{51}$$

does not factorize into two independent $\pm$ partition functions. But since the Chern class only depends on the global properties of the gauge field, the local *i.e.* perturbative fluctuations of the fields A^+ and A^- are independent. This implies that in the perturbative sector parametrized by Q we indeed have a *local* SU(N)$_+\times$SU(N)$_-$ gauge invariance.

In order to eliminate this SU(N)$_+\times$SU(N)$_-$ gauge invariance, we need a BRST operator for both SU(N): In addition of the BRST operator (26) which eliminates the SU(N)$_-$ gauge invariance

$$\Omega_{YM}^- = \mathbf{u}\mathcal{G}^- + \frac{1}{2}\mathbf{v}[\mathbf{u},\mathbf{u}] + h\bar{\mathbf{u}} \tag{52}$$

where the Gauss law operator $\mathcal{G}^-$ is now

$$\mathcal{G}^- = D_\mu^- E_\mu^- + [\varphi,\pi] + [\bar{\varphi},\bar{\pi}] + [\beta,\gamma] + [\psi_\mu,\mathcal{X}_\mu] + [\bar{\psi}_{\mu\nu},\bar{\mathcal{X}}_{\mu\nu}] + [b_{\mu\nu},c_{\mu\nu}] \tag{53}$$

we also introduce the following nilpotent BRST operator for the SU(N)$_+$ gauge group

$$\Omega_{YM}^+ = \mathbf{c}D_\mu^+ E_\mu^+ + \frac{1}{2}\mathbf{b}[\mathbf{c},\mathbf{c}] + t\bar{\mathbf{c}} \tag{54}$$

We then add the corresponding gauge fixing terms to the action in (51),

$$S_{YM}(A^+) + S_{TOP}(A^-) = \int \frac{1}{4}F_{\mu\nu}^2(A^+) + \sum_{i=1}^4 \alpha_i\{\Omega_{TOP}^-, \Psi_i\} + \{\Omega_{YM}^+, \Psi^+\} + \{\Omega_{YM}^-, \Psi^-\} \tag{55}$$

Here Ω_{TOP}^- and Ω_{YM}^- depend only on A^-, E^- while Ω_{YM}^+ depends only on A^+, E^+. Hence Ω_{YM}^+ anticommutes with both Ω_{TOP} and Ω_{YM}^- and the ensuing path integral is invariant under local variations of the gauge fermions $\Psi_i(A^-)$, $\Psi^\pm(A^\pm)$.

Notice that since the $\pm$-theories couple only by nonperturbative boundary terms, in perturbation theory the quantum theory determined by (55) coincides with the standard Yang-Mills perturbation theory, and in particular the perturbative high energy

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β-functions of these two theories coincide. Only when nonperturbative instanton effects become relevant, will (55) deviate from standard Yang-Mills theory. Notice also that even though our boundary condition (50) breaks cluster decomposition nonperturbatively, perturbative Green's functions do obey cluster decomposition. This is consistent with the ultraviolet asymptotic freedom which implies that in the high energy limit the gauge fields are physical degrees of freedom. This is also consistent with the (expected) infrared confinement of Yang-Mills theory, which implies that in the infrared limit we can not directly associate the gauge fields with physical degrees of freedom.

Finally we observe that if we select $\tau = \frac{1}{2}$ and $\alpha_1 = 4$ as in (32), we find a particularly convenient representation of the Yang-Mills part of the action (55): We first obtain

$$-\frac{1}{4}F_{\mu\nu}^2(A^+) \ - \ \frac{1}{4}F_{\mu\nu}^2(A^-) \ \rightarrow \ -\frac{1}{4}F_{\mu\nu}^2(A+\frac{1}{2}Q) \ - \ \frac{1}{4}F_{\mu\nu}^2(A-\frac{1}{2}Q) \qquad (56)$$

but since this is an even function in the fluctuation field Q_μ, the linear and cubic terms in Q_μ disappear. In particular the term which is linear in Q_μ and proportional to the Yang-Mills equation of motion will be absent. If we rescale Q by a factor of 2, we then have explicitly (excluding the gauge fixing terms)

$$-S_{YM} - S_{TOP} \ = \ -\frac{1}{2}F_{\mu\nu}^2 \ - \ \frac{1}{2}G_{\mu\nu}^2 \ - \ F_{\mu\nu}[Q_\mu, Q_\nu] \ - \ \frac{1}{2}[Q_\mu, Q_\nu]^2 \ - \ \frac{1}{2}\phi D_\mu^2 \lambda \ + \ i\eta D_\mu \psi_\mu$$

$$- \ i \, D_\mu \psi_\nu \mathcal{X}_{\mu\nu} \ + \ \frac{i}{8}\phi[\mathcal{X}_{\mu\nu}, \mathcal{X}_{\mu\nu}] \ + \ \frac{i}{2}\lambda[\psi_\mu, \psi_\mu] \ + \ \frac{i}{2}\phi[\eta, \eta] \ + \ \frac{1}{8}[\phi, \lambda]^2 \ + \ 4\pi^2 N \quad (57)$$

Here the explicit covariant derivatives are now *w.r.t.* the connection $A - Q$, and the notation coincides with that in [4]. The $4\pi^2 N$ denotes the second Chern class that appears in (33).

5   A gauge invariant mass scale

By construction, the action (55), (57) has a local $SU(N)_+ \times SU(N)_-$ gauge invariance. However, the physically relevant gauge invariance is the $SU(N)$ symmetry of our original Yang-Mills theory. This suggests, that physically interesting consequences such as the possibility to introduce an explicit gauge invariant mass scale might emerge, if we allow the explicit breaking of the $SU(N)_+ \times SU(N)_-$ symmetry into the original $SU(N)$. From the point of view of our original Yang-Mills theory we may then view the ensuing $SU(N)$

gauge theory, obtained by explicitly breaking the $SU(N)_+ \times SU(N)_-$ gauge symmetry into a $SU(N)$, as a natural extension of the original theory which may have a number of physically desirable features.

Notice that in general an explicit gauge symmetry breaking will also break the topological symmetry determined by Ω_{TOP}^-. As a consequence the topological action will depend nontrivially on the parameters α_i, which now become coupling constants in our theory.

In order to explicitly break the extra $SU(N)$ gauge symmetry, we observe that under the diagonal $SU(N)$ gauge transformations the linear combination

$$\mathcal{A}_\mu = \frac{1}{2}(A_\mu^+ + A_\mu^-) = A_\mu + \frac{1}{2}(1 - 2\tau)Q_\mu \xrightarrow{\tau = \frac{1}{2}} A_\mu \tag{58}$$

transforms inhomogeneously like a gauge field, while the linear combinations

$$\Phi_\mu = A^+ - A^- = Q_\mu \tag{59}$$

transform homogeneously, like a Higgs field. This suggests, that we should consider to break the extra $SU(N)$ symmetry by introducing $e.g.$ a gauge invariant mass term for the "Higgs vector" Φ_μ. Consequently, instead of the original $A_\mu^\pm$ fields it may be more convenient to represent our action using the fields (58), (59).

In order to explicitly break the $SU(N) \times SU(N)$ invariance, we first consider the BRST gauge fixing of these symmetries in our action. For this we define the following two functionals,

$$\begin{aligned} \Psi^+ &= \bar{\mathbf{v}}(\xi h + \partial_\mu A_\mu^-) \\ \Psi^- &= \bar{\mathbf{b}}(\alpha t + D_\mu^- \Phi_\mu) \end{aligned}$$

and consider the ensuing BRST gauge fixing terms

$$\{\Omega_{YM}^+, \Psi^+\} + \{\Omega_{YM}^-, \Psi^-\}$$

in the action (55). After we have eliminated the auxiliary fields t^a, h^a from the path integral by Gaussian integration we find the following contribution,

$$\frac{1}{4\xi}(\partial_\mu A_\mu^-)^2 + \mathbf{u}D_\mu^- \partial_\mu \bar{\mathbf{v}} + \frac{1}{4\alpha}(D_\mu^- \Phi_\mu)^2 + \mathbf{c}D_\mu^+ D_\mu^- \bar{\mathbf{b}} + \bar{\mathbf{b}}D_\mu^+ D_\mu^- \mathbf{u}$$

Here we identify the first two terms as the familiar gauge fixing and Faddeev-Popov ghost terms in the covariant R_ξ-gauge for A_μ^-. In addition, when $\alpha \to 0$ we find from the third term the condition

$$D_\mu^- \Phi_\mu = 0 \tag{60}$$

We can interpret this condition as follows: The $\text{SU(N)}_+ \times \text{SU(N)}_-$ gauge invariance corresponds to the original background formalism gauge invariance in (6). The condition (60) fixes the gauge invariance of the fluctuation field $\Phi_\mu \sim Q_\mu$ with respect to the classical field A_μ ($\sim A_\mu$ for $\tau = 0$), while the R_ξ gauge condition fixes the remaining gauge invariance of the classical field A_μ.

The condition (60) has also an alternative interpretation. It can be viewed as a unitarity condition for the fluctuation field $Q_\mu \sim \Phi_\mu$, since it eliminates the negative metric time component Φ_0 in a covariant fashion.

The present construction suggests that we explicitly break the $\text{SU(N)}_+ \times \text{SU(N)}_-$ gauge invariance into the SU(N) invariance of our original Yang-Mills theory as follows: Since the field Φ_μ transforms like a Higgs field (59) under gauge transformations and since (60) eliminates the unitarity violating time component Φ_0, we may view Φ_μ as a vector analog of the conventional scalar Higgs. Consequently we can explicitly break the $\text{SU(N)}_+ \times \text{SU(N)}_-$ into the diagonal SU(N) by adding the following "Higgs mass" to the action (57),

$$S_m(\Phi) = m^2 \Phi_\mu^2 \tag{61}$$

Our action is then

$$S = S_{YM} + S_{TOP} + S_m \tag{62}$$

Since the $\text{SU(N)}_+ \times \text{SU(N)}_-$ gauge invariance has now been broken, we have also lost the BRST invariances under (52) and (54). Hence we must construct a new BRST operator for the remaining SU(N) gauge symmetry. This BRST invariance must also preserve the unitarity condition (60). Since this unitarity condition is gauge covariant, we conclude that the relevant nilpotent BRST operator is

$$\Omega_{diag} = \mathbf{c}^a(\mathcal{G}^a + f^{abc}\mathbf{u}^b\mathbf{v}^c) + \frac{1}{2}f^{abc}\mathbf{c}^a\mathbf{c}^b\mathbf{b}^c + \mathbf{u}^a D_\mu^{ab}\Phi_\mu^b + t^a\bar{\mathbf{c}}^a + h^a\bar{\mathbf{u}}^a \tag{63}$$

and our gauge fixed action is

$$S = S_{YM} + S_{TOP} + S_m + \{\Omega_{diag}, \Psi\} \tag{64}$$

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Indeed, consider the gauge fermion

$$\Psi \;=\; \Psi_1 \;+\; \Psi_2 \;=\; \alpha\left(\mathbf{v}^a D_\mu^{ab}\Phi_\mu^b \;+\; h^a\bar{\mathbf{v}}^a\right) \;+\; \bar{\mathbf{b}}^a(\partial_\mu A_\mu^a - \frac{1}{4\xi}t^a) \tag{65}$$

Here α, ξ are gauge parameters and the Ψ-invariance ensures that the corresponding path integral is independent of these parameters. After we have eliminated t^a by Gaussian integration we find the following gauge fixed version of (64)

$$
\begin{aligned}
-S \;=\; & -\frac{1}{2}F_{\mu\nu}^2 \;-\; \frac{1}{2}G_{\mu\nu}^2 \;-\; F_{\mu\nu}[\Phi_\mu,\Phi_\nu] \;-\; m^2\Phi_\mu^2 \;-\; \frac{1}{2}[\Phi_\mu,\Phi_\nu]^2 \;-\; \frac{1}{2}\phi D_\mu^2\lambda \\
& + i\eta D_\mu\psi_\mu \;-\; i\,D_\mu\psi_\nu\mathcal{X}_{\mu\nu} \;+\; \frac{i}{8}\phi[\mathcal{X}_{\mu\nu},\mathcal{X}_{\mu\nu}] \;+\; \frac{i}{2}\lambda[\psi_\mu,\psi_\mu] \;+\; \frac{i}{2}\phi[\eta,\eta] \;+\; \frac{1}{8}[\phi,\lambda]^2 \\
& -\; \alpha(D_\mu\Phi_\mu)^2 \;-\; \xi(\partial_\mu A_\mu)^2 \;+\; \bar{\mathbf{b}}^a\partial_\mu D_\mu^{ab}\mathbf{c}^b \;+\; 4\pi^2 N
\end{aligned}
\tag{66}
$$

The $\alpha \to \infty$ limit then yields the unitarity condition

$$D_\mu^{ab}\Phi_\mu^b \;=\; \partial_\mu\Phi_\mu^a + f^{acb}A_\mu^c\Phi_\mu^b \;=\; 0 \tag{67}$$

and the $\mathbf{b}$, $\bar{\mathbf{c}}$ ghost term coincides with the familiar Faddeev-Popov ghost term in the covariant R_ξ gauge for A_μ.

The gauge fixed action (66) describes the coupling of a massive vector field Φ_μ to a SU(N) gauge field and the various ghost fields of the topological Yang-Mills theory. Since the mass term for Φ_μ breaks the topological BRST invariance under Ω_{TOP} in (55), these topological ghost fields become physical. However, in the following section we shall argue that the corresponding quantum theory is nonetheless unitary in the physical subspace defined by demanding BRST invariance under (63). Here we shall argue that (66) is also power counting renormalizable. For this, we observe that all interactions and ghost propagators are power counting renormalizable. Consequently it is sufficient to verify that the $A-\Phi$ propagators are also consistent with power counting renormalizability *i.e.* that these propagators vanish like $\mathcal{O}(p^{-2})$ in the $p^2 \to \infty$ limit. Indeed, if for simplicity we rescale

$$A_\mu \;\to\; \frac{1}{\sqrt{2}}A_\mu$$

$$\Phi_\mu \;\to\; \frac{1}{\sqrt{2}}\Phi_\mu$$

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we find for the propagators

$$
\begin{aligned}
\Delta_{\mu\nu}^{AA}(p) &= \frac{1}{p^2}\left(\delta_{\mu\nu} - (1 - \frac{1}{\xi})\frac{p_\mu p_\nu}{p^2}\right) \\
\Delta_{\mu\nu}^{\Phi\Phi}(p) &= \left(\delta_{\mu\nu} - (1 - \frac{1}{\alpha})\frac{p_\mu p_\nu}{p^2 + \frac{1}{\alpha}m^2}\right)\frac{1}{p^2 + m^2}
\end{aligned}
\tag{68}
$$

Here the $< AA >$-propagator has the standard form of a massless gauge vector propagator in the covariant R_ξ-gauge. Similarly, the $< \Phi\Phi >$-propagator has the standard form of a massive gauge vector propagator in the covariant R_α-gauge. Since both propagators vanish like $\mathcal{O}(p^{-2})$ for large values of p^2 we then conclude that our action is (at least) power counting renormalizable, and the physical spectrum contains both a massless gauge field A_μ and a massive gauge field Φ_μ.

We conclude this section by noting, that (61) is not the most general SU(N) invariant Φ_μ-dependent term that one can introduce. However, (61) is particularly interesting, since it introduces a power counting renormalizable mass scale in a minimal and covariant fashion. More generally, one may consider adding for example

$$
S_\Phi(\Phi) = \frac{1}{2}(D_\mu\Phi_\nu)^2 + \frac{m^2}{2}\Phi_\mu^2 + \frac{g}{4}(\Phi_\mu^2)^2
\tag{69}
$$

This coincides with the conventional form of a Higgs action with four species of Higgs fields Φ_μ. This action is also the most general one which is consistent with a twisted version of a Lorentz transformation where Φ_μ transforms as a scalar instead of as a vector, even though this twisted Lorentz invariance is broken by the additional $\Phi_\mu \sim Q_\mu$ dependent terms in (57). Indeed, the more general choice (69) can be motivated by the *no-go* theorem derived in [18], which states that if we are interested in introducing a renormalizable mass for a gauge vector, a version of the Higgs field is unavoidable. (However, here we have also found that for power counting renormalizability it is sufficient to consider (61) only. This is due to the additional Q_μ dependence which is present in our action.)

Using (69), we can construct the following interesting and manifestly unitary alternative to our action (66). For this we first explicitly eliminate $\Phi_0 \sim Q_0$: Since A^+ and A^- are independent gauge fields, we can perform independent SU$_\pm$(N) gauge rotations in (57) to gauge transform both fields to the temporal gauge $A_0^\pm = 0$ before we explicitly

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break these gauge symmetries into the diagonal SU(N). In terms of our original fields, this means that we set

$$A_0 \;=\; Q_0 \;=\; 0$$

and in particular $\Phi_0 = 0$. In this gauge we can then explicitly break the remaining time-independent SU$_+$(N)×SU$_-$(N) gauge invariance into the diagonal SU(N), by introducing the $\Phi_0 = 0$ version of (69)

$$S_\Phi \;=\; \frac{1}{2}(D_\mu \Phi_i)^2 \;+\; \frac{m^2}{2}\Phi_i^2 \;+\; \frac{g}{4}(\Phi_i^2)^2$$

We now have a manifestly unitary theory, albeit in a gauge which is not manifestly Lorentz-invariant. However, this "gauge fixed" version of our theory is particularly interesting, since we now have three "scalar fields" Φ_i in the adjoint representation of the gauge group. In analogy with standard Yang-Mills-Higgs systems, we then expect that also in the present case the classical equations of motion admit finite energy magnetic monopole solutions provided we select m^2 and g so that we have a potential $V(\Phi)$ that exhibits spontaneous symmetry breaking. Indeed, since these solutions are time independent, by a gauge transformation we can similarly eliminate one of the space components of Φ_i and we are left with only two "scalar fields" Φ_z and $\Phi_{\bar{z}}$.

Notice that since the "Higgs field" Φ_i by construction obeys a trivial asymptotic boundary condition, we should only consider monopole configurations that are consistent with this boundary condition. This means that there must be an equal number of monopoles and antimonopoles, and in particular we can not have any "free" monopoles. In this sense the present construction has certain aspects that are reminiscent of the picture of confinement introduced in [2].

6   Unitarity and N=2 supersymmetry

Since the action (66) fails to be invariant under the BRST operator (48), there is nothing a priori that would prevent the ghost fields of the topological Yang-Mills theory from appearing in the physical spectrum. But since these ghosts violate the spin-statistics theorem, this means that unitarity is in peril. We shall now propose that the quantum theory of (66) is nevertheless unitary, by reformulating it in terms of manifestly unitary

variables. For this we use the fact that the topological Yang-Mills theory can be viewed as a twisted version of the N=2 supersymmetric Yang-Mills theory [4], obtained by re-interpreting the action of the Lorentz-group.

The (Minkowski space) action of the minimal SU(N) invariant N=2 supersymmetric Yang-Mills theory is

$$-S_{N=2} = -\frac{1}{4}F_{\mu\nu}^2 - D_\mu B D_\mu \bar{B} - i\bar{\lambda}_i \bar{\sigma}_\mu D_\mu \lambda^i - \frac{1}{\sqrt{2}}B[\bar{\lambda}_i, \bar{\lambda}^i] + \frac{1}{\sqrt{2}}\bar{B}[\lambda_i, \lambda^i] + \frac{1}{2}[B, \bar{B}]^2 \quad (70)$$

By comparing the actions (33) and (70) term-by-term, we observe an obvious similarity. Indeed, since the Lorentz algebra SO(3,1) is related to SO(4)~SU(2)×SU(2) and since the action (70) has an internal SU(2)^N symmetry, we can redefine the action of the Lorentz group by twisting the fields. This means that we introduce an invertible change of variables between (33) and (70)

$$(B, \bar{B}, \lambda_i, \bar{\lambda}_i, A_\mu) \rightarrow (\phi, \lambda, \eta, \psi_\mu, \mathcal{X}_{\mu\nu}, A_\mu)$$

which maps (70) into (33) and *vice versa*. This change of variables is defined by

$$
\begin{aligned}
B &= -i\sqrt{2}\phi \\
\bar{B} &= -\frac{i}{\sqrt{8}}\lambda \\
\lambda_{\alpha i} &= -\sigma_{\alpha i}^\mu \psi^\mu \\
\bar{\lambda}_{\dot{\alpha} i} &= -\frac{1}{2}\epsilon_{\dot{\alpha} i}\eta + \frac{1}{4}\bar{\sigma}_{\dot{\alpha} i}^{\mu\nu}\mathcal{X}_{\mu\nu}
\end{aligned}
\quad (71)
$$

The inverse transformation is

$$
\begin{aligned}
\phi &= \frac{i}{\sqrt{2}}B \\
\lambda &= i\sqrt{8}\bar{B} \\
\eta &= \bar{\lambda}_i{}^i \\
\psi_\mu &= \frac{1}{2}\sigma_\mu^{\alpha i}\lambda_{\alpha i} \\
\mathcal{X}_{\mu\nu} &= 2\bar{\sigma}_{\mu\nu}^{\dot{\alpha} i}\bar{\lambda}_{\dot{\alpha} i}
\end{aligned}
\quad (72)
$$

If we substitute (71) in the N=2 action (70) (modulo analytic continuation to the Euclidean space and the topological $F\tilde{F}$ term) we find

$$-S = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\phi D_\mu^2 \lambda + i\eta D_\mu \psi_\mu + i\mathcal{X}_{\mu\nu}D_\mu \psi_\nu + \frac{i}{8}\phi[\mathcal{X}_{\mu\nu}, \mathcal{X}_{\mu\nu}]$$

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$$+ \frac{i}{2}\lambda[\psi_\mu, \psi_\mu] \; + \; \frac{i}{2}\phi[\eta, \eta] \; + \; \frac{1}{8}[\phi, \lambda]^2 \tag{73}$$

which is the action (33) of topological Yang-Mills theory. Since this change of variables has a trivial Jacobian in the path integral, we conclude that the ensuing partition functions coincide. Only the interpretation of these two theories is different.

Alternatively, we can introduce a change of variables which relates the N=2 theory to the selfdual topological Yang-Mills theory instead of the anti-selfdual one.

Twisting also has an effect on the supersymmetry algebra: For the N=2 theory we have the following N=2 supersymmetry algebra,

$$\begin{aligned}
\{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\} &= 2\delta^i{}_j \sigma^\mu{}_{\alpha\dot\beta} P_\mu \\
\{Q_\alpha{}^i, Q_\beta{}^j\} &= \epsilon_{\alpha\beta}\epsilon^{ij} Z \\
\{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\} &= -\epsilon_{\dot\alpha\dot\beta}\epsilon_{ij} Z^\star
\end{aligned} \tag{74}$$

where $Z \sim a n_e + a_D n_m$ is the central charge. Using our change of variables (71), (72) we then find, that the BRST operator (25) of the topological Yang-Mills theory is related to the N=2 supersymmetry generators of (70) by

$$\Omega \; = \; \epsilon^{\dot\alpha i}\bar{Q}_{\dot\alpha i} \tag{75}$$

Furthermore, since

$$\{Q_1{}^2, \bar{Q}_{\dot 1 2}\} \; = \; -2(H - P_3) \tag{76}$$

and

$$\{Q_2{}^1, \bar{Q}_{\dot 2 1}\} \; = \; -2(H + P_3) \tag{77}$$

where H is the Hamiltonian of (70), combining (76) and (77) we get

$$H \; = \; -\frac{1}{4}\{\bar{Q}_{\dot 1 2} - \bar{Q}_{\dot 2 1}, Q_1{}^2 - Q_2{}^1\} \; = \; \{\Omega, \Psi\}$$

which implies that the gauge fermion Ψ that yields the action (73) of topological Yang-Mills theory is

$$\Psi \; = \; \frac{1}{4}(Q_2{}^1 - Q_1{}^2)$$

Finally, if we define the following two operators

$$\begin{aligned}
Q^\mu &= \sigma^\mu_{\alpha i}\epsilon^{\alpha\beta} Q_\beta{}^i \\
D^{\mu\nu} &= \bar{\sigma}^{\mu\nu\dot\alpha}_{\dot\beta}\epsilon^{\dot\beta i}\bar{Q}_{\dot\alpha i}
\end{aligned}$$

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and introduce the antiself-dual projection operator

$$\mathcal{P}_-{}^{\mu\nu\rho\sigma} \;=\; \frac{1}{4}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}\right)$$

we find that in terms of the twisted variables the supersymmetry algebra (74) becomes

$$
\begin{aligned}
\{\Omega, \Omega\} &= -2Z^\star \\
\{\Omega, Q^\mu\} &= 4P^\mu \\
\{\Omega, D^{\mu\nu}\} &= 0 \\
\{Q^\mu, Q^\nu\} &= 2\eta^{\mu\nu}Z \\
\{Q^\mu, D^{\rho\sigma}\} &= 8P_\nu \mathcal{P}_-{}^{\rho\sigma\mu\nu} \\
\{D^{\mu\nu}, D^{\rho\sigma}\} &= -2Z^\star \mathcal{P}_-{}^{\mu\nu\rho\sigma}
\end{aligned}
$$

Curiously, we find that if the central charge $Z \sim a n_e + a_D n_m$ is nonvanishing, the BRST operator becomes non-nilpotent. Since nilpotency is an essential property of the BRST operator, this suggests an interesting possibility to break the BRST symmetry in a dynamical fashion. It would be interesting to understand, whether such a dynamical BRST symmetry breaking could provide an alternative to our explicit breaking of the BRST symmetries.

We now return to the action (66) that we introduced in the previous section. In terms of the N=2 variables (71) we find that this action becomes

$$-S \;=\; -\frac{1}{2}F_{\mu\nu}^2 \;-\; \frac{1}{2}G_{\mu\nu}^2 \;-\; F_{\mu\nu}[\Phi_\mu, \Phi_\nu] \;-\; m^2\Phi_\mu^2 \;-\; \frac{1}{2}[\Phi_\mu, \Phi_\nu]^2 \;-\; D_\mu B D_\mu \bar{B}$$

$$-\; i\bar{\lambda}_i \bar{\sigma}_\mu D_\mu \lambda^i \;+\; \frac{1}{2}[B, \bar{B}]^2 \;-\; \frac{1}{\sqrt{2}}B[\bar{\lambda}_i, \bar{\lambda}^i] \;+\; \frac{1}{\sqrt{2}}\bar{B}[\lambda_i, \lambda^i] \;+\; 4\pi^2 N \;+\; \{\Omega_{diag}, \Psi\} \quad (78)$$

where the BRST operator is defined by (63), and it fixes the SU(N) gauge invariance of (78) and also eliminates the nonunitary time component of Φ_μ. This action is consistent with the spin-statistics theorem, and recalling our discussion from the previous section it also appears to be unitarity and power counting renormalizable. Thus we conclude that this massive version of Yang-Mills theory appears to define a consistent quantum field theory in four dimensions. In particular, since (78) is obtained from the standard Yang-Mills theory simply by adding the mass term for the vector field Φ_μ, it can be viewed as a natural massive generalization of the standard Yang-Mills theory. Furthermore, since

the N=2 supersymmetric Yang-Mills theory which is contained in (78) has a very rich structure and in particular confines [1], the present extension may also imply physically interesting consequences for the ordinary Yang-Mills theory.

Notice that besides (61) there are also additional gauge invariant, power-counting renormalizable and apparently unitary mass terms that we can introduce. For example, following [1] we could introduce a mass scale in (70) by adding a mass term for the scalar superfield. This mass term breaks the N=2 supersymmetry explicitly into an N=1 supersymmetry, and as argued in [1] in the context of the N=2 supersymmetry the resulting theory exhibits both color and quark confinement. In the present case, in the absence of the explicit $SU(N)_+ \times SU(N)_-$ breaking terms such as (61) we then have an effective theory which couples this confining N=1 theory to the original Yang-Mills theory only by the non-perturbative fluctuations.

We conclude this section by observing, that since we are dealing with an effective theory obtained by summing over anti-selfdual configurations in (51), there is no a priori reason why our theory should exhibit perturbative renormalizability. Unlike unitarity, perturbative renormalizability is in general not expected to survive in an effective theory. Rather, our arguments for renormalizability should be viewed as a curiosity, that may point out new possibilities to desribe massive gauge vectors without the Higgs effect. Furthermore, we recall that (51) is only an approximation to the original Yang-Mills path integral, and we should also account for the highly nonlocal phase factors in (21). These phase factors break both perturbative renormalizability and manifest locality. However, since these phase factors represent an additive structure in our theory, we do expect that the results derived in the present section, in particular the N=2 structure, are also present in the original, exact version (2) of the Yang-Mills partition function. Finally, since we are ultimately interested in the infrared limit it is feasible to conjecture that in this limit these highly complicated phase factors become irrelevant operators.

7 Quantum Morse theory and cohomology

In the previous construction we have restricted our attention to the anti-selfdual solutions, approximating the partition function (2) by (17). We shall now consider the improvement (13) that we obtain when we sum over all possible solutions to the Yang-

Mills equations,

$$Z_{YM} \approx \int [dQ][dA] \, \delta(DF) \det ||\frac{\delta DF}{\delta A}|| \exp\{-\int \frac{1}{4}F_{\mu\nu}^2(A+Q)\} \tag{79}$$

As we have explained in Section 2, we can formally interpret this as an equivariant generalization of the Euler character $\mathcal{X}(\mathcal{A}/\mathcal{G})$ on the gauge orbit space,

$$\mathcal{X}(\mathcal{A}/\mathcal{G}) \approx \int [dA]\delta(DF) \det ||\frac{\delta DF}{\delta A}||$$

$$= \sum_{DF=0} sign \det ||\frac{\delta DF}{\delta A}|| \tag{80}$$

In order to utilize this interpretation to study (79), we first develop certain formal aspects. For this we consider a generic D-dimensional quantum field theory defined by an action $S(\phi^a)$, where $\{\phi^a\}$ are fields that take values on some configuration space which in the case of Yang-Mills theory is $\mathcal{A}/\mathcal{G}$. For simplicity we shall assume that $S(\phi^a)$ has the same formal properties as a nondegenerate Morse function.

Formally, according to the Poincáre-Hopf theorem the Euler character of the $\{\phi^a\}$ field space can be represented as

$$\mathcal{X}(\phi) = \sum_{\delta S=0} sign \det ||\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}|| = \int [d\phi]\delta\left(\frac{\delta S}{\delta\phi^a}\right) \det ||\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}||$$

$$= \int [d\phi][d\pi][d\psi][d\bar{\mathcal{P}}] \exp\{i\int \pi^a\frac{\delta S}{\delta\phi^a} + \psi^a\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bar{\mathcal{P}}^b\} \tag{81}$$

and a priori this is independent of the Morse function $S(\phi)$. Here we have introduced one commuting (π^a) and two anticommuting $(\psi^a, \bar{\mathcal{P}}^a)$ auxiliary variables, to exponentiate the δ-function and the determinant respectively. The action in (81)

$$S_{eff} = \int \pi^a\frac{\delta S}{\delta\phi^a} + \psi^a\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bar{\mathcal{P}}^b \tag{82}$$

has the following nilpotent supersymmetry

$$\begin{aligned}
\Omega\phi^a &= \psi^a \\
\Omega\psi^a &= 0 \\
\Omega\bar{\mathcal{P}}^a &= \pi^a \\
\Omega\pi^a &= 0
\end{aligned} \tag{83}$$

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so that we can represent Ω by

$$\Omega = \psi^a \frac{\delta}{\delta\phi^a} + \pi^a \frac{\delta}{\delta\bar{\mathcal{P}}^a} \tag{84}$$

and clearly

$$\Omega^2 = 0$$

We identify (83) as the Parisi-Sourlas supersymmetry [5], when realized on a scalar superfield in the Parisi-Sourlas superspace. In addition of the space coordinates x, this superspace has two anticommuting coordinates θ and $\bar{\theta}$

$$\theta^2 = \bar{\theta}^2 = \theta\bar{\theta} + \bar{\theta}\theta = 0$$

The scalar superfield is

$$\Phi^a(x, \theta, \bar{\theta}) = \phi^a + \theta\psi^a - \bar{\theta}\bar{\mathcal{P}}^a + \theta\bar{\theta}\pi^a \tag{85}$$

and the supersymmetry (83) can be identified with the translation in the θ direction of the superspace,

$$\Omega = \int \partial_\theta \Phi^a \frac{\delta}{\delta\Phi^a} \sim \partial_\theta \tag{86}$$

Using (85) we can write the action in (81) as

$$\int dx \left(\pi^a \frac{\delta S}{\delta\phi^a} + \psi^a \frac{\delta^2 S}{\delta\phi^a\delta\phi^b} \bar{\mathcal{P}}^b \right) = \int dx d\bar{\theta} d\theta \; S(\Phi)$$

Hence, as a function of the superfield Φ^a the action (82) has the same functional form as our original action $S(\phi)$. Furthermore, if we introduce the following functional

$$\Psi = \bar{\mathcal{P}}^a \frac{\delta S}{\delta\phi^a} \tag{87}$$

we can represent (82) as closed under the supersymmetry operator (84)

$$\int \pi^a \frac{\delta S}{\delta\phi^a} + \psi^a \frac{\delta^2 S}{\delta\phi^a\delta\phi^b} \bar{\mathcal{P}}^b = \int \Omega\Psi \tag{88}$$

Consequently the path integral (81) is of the standard cohomological form (23),

$$\mathcal{X}(\phi) = \int [d\phi][d\pi][d\bar{\mathcal{P}}][d\psi] \exp\{i \int \Omega\Psi\} \tag{89}$$

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and in particular it is invariant under local variations of the gauge fermion Ψ. Notice that this also ensures that the Euler character (81) is indeed independent of the local details of the "Morse functional" $S(\phi)$.

We shall now apply the Ψ-invariance of (89) to generalize (87) to

$$\Psi \;=\; \bar{\mathcal{P}}^a \big(\frac{\delta S}{\delta \phi^a} + \frac{\kappa}{2} \pi^a \big) \tag{90}$$

where κ is a parameter. For the action this yields

$$\int \Omega \Psi \;=\; \int \pi^a \frac{\delta S}{\delta \phi^a} \;+\; \psi^a \frac{\delta^2 S}{\delta \phi^a \delta \phi^b} \bar{\mathcal{P}}^b \;-\; \frac{\kappa}{2} \pi^2$$

and in terms of the superfield Φ we get

$$\int \Omega \Psi \;=\; \int dx d\bar{\theta} d\theta \Big\{ S(\Phi) - \frac{\kappa}{2} \Phi^a \partial_{\bar{\theta}} \partial_\theta \Phi^a \Big\} \tag{91}$$

If the original action $S(\phi)$ has the standard functional form

$$S(\phi) \;=\; \frac{1}{2} \phi^a (-\square) \phi^a \;+\; V(\phi) \tag{92}$$

we then conclude that the superspace action can be represented in the corresponding superspace form

$$S(\Phi) \;=\; \frac{1}{2} \Phi^a (-\square - \kappa \partial_{\bar{\theta}} \partial_\theta) \Phi^a \;+\; V(\Phi) \tag{93}$$

This is the standard Parisi-Sourlas action for a scalar field theory that has been extensively investigated in [5]. In particular, it has been established that in perturbation theory the superspace quantum field theory determined by (93) coincides diagram-by-diagram with the quantum field theory of (92), but in two less space-time dimensions. This $D \to D\text{-}2$ dimensional reduction is a consequence of the negative dimensionality of the anticommuting coordinates [5]. The dimensional transformation from the D dimensional coupling constants *etc.* to their D-2 dimensional counterparts is determined by the parameter κ which has the dimensions

$$[\kappa] \;\propto\; m^2$$

when we define the anticommuting variables θ and $\bar{\theta}$ to be dimensionless. The overall numerical scale of κ is undetermined, and can be changed by redefining the normalization of the θ, $\bar{\theta}$ integral.

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As explained in [5], the superspace quantum theory can also be interpreted in terms of a stochastic differential equation. For this we introduce an additional variable h and write the path integral (89), (90) in the following equivalent form

$$\mathcal{X}(\phi) = \int [\frac{1}{2\sqrt{\kappa}} dh][d\phi]\delta(\frac{\delta S}{\delta \phi^a} - h^a) \det ||\frac{\delta^2 S}{\delta \phi^a \delta \phi^b}|| \exp\{-\int \frac{1}{4\kappa} h^2\}$$

$$= \int [\frac{1}{2\sqrt{\kappa}} dh] \exp\{-\int \frac{1}{4\kappa} h^2\} \sum_{\frac{\delta S}{\delta \phi^a} = h^a} sign \det ||\frac{\delta^2 S}{\delta \phi^a \delta \phi^b}|| \qquad (94)$$

This has the interpretation of averaging classical solutions to the stochastic differential equation

$$\frac{\delta S}{\delta \phi^a} = -\Box \phi^a + \partial_a V(\phi) = h^a \qquad (95)$$

over the external Gaussian random source h. Notice that as a consequence of the Ψ invariance the integral (94) is actually independent of κ, and if we take the $\kappa \to 0$ limit and recall the Gaussian definition of a δ-function we find that (94) reduces to (81). This is fully consistent with the fact that the Euler character is independent of the Morse functional. Indeed, from the Morse theory point of view

$$S_h(\phi) = S(\phi) + h^a \phi^a$$

is simply another (nondegenerate) Morse functional. (Here we assume that the $h^a \phi^a$ term is a small perturbation in the sense described in [19].)

Finally, we conclude this section by deriving the Gauss-Bonnet-Chern theorem that represents $\mathcal{X}(\phi)$ in terms of the curvature two-form on the configuration space $\{\phi^a\}$. (Here we assume that ϕ^a has a nontrivial topology; If the configuration space is a flat Euclidean manifold, see [19]). For this we introduce a canonical transformation, determined by conjugating Ω

$$\Omega \rightarrow e^{-U} \Omega e^U$$

We select

$$U = -\Gamma^a_{bc} \psi^c \bar{\mathcal{P}}_a \lambda^b$$

where we have introduced the conjugate variable

$$\{\pi^a, \lambda^b\} = -\delta^{ab}$$

and $\Gamma^a_{bc}(\phi)$ are components of a connection on the configuration space $\{\phi^a\}$. For the conjugated Ω we find the following transformation laws,

$$
\begin{aligned}
\Omega\phi^a &= \psi^a \\
\Omega\psi^a &= 0 \\
\Omega\bar{\mathcal{P}}_a &= \pi_a + \Gamma^c_{ab}\psi^b\bar{\mathcal{P}}_c \\
\Omega\pi_a &= \Gamma^c_{ab}\pi_c\psi^b - \frac{1}{2}R^c{}_{adb}\psi^b\psi^d\bar{\mathcal{P}}_c
\end{aligned}
\tag{96}
$$

which we identify as the familiar transformation law of the standard (N=1) de Rham supersymmetric quantum mechanics [12]. Indeed, if we assume that the connection Γ^a_{bc} is metric and use the metric tensor $g_{ab}(\phi)$ to define

$$
\Psi = g^{ab}\pi_a\bar{\mathcal{P}}_b
$$

we immediately find that the corresponding path integral (89) evaluates to

$$
\mathcal{X}(\phi) = \int[d\phi][d\psi]\ \text{Pf}[\frac{1}{2}R^a{}_{bcd}\psi^c\psi^d]
\tag{97}
$$

This is the formal (functional) Euler class of the configuration space $\{\phi^a\}$, and establishes that the Gauss-Bonnet-Chern representation of the Euler character indeed coincides with the Poincaré-Hopf representation (81), (94).

8 All Solutions Approximation

We shall now study the approximation (13) to the Yang-Mills partition function, using the formalism we have developed in the previous section. We start by considering first the formal Poincaré-Hopf representation of the Euler character on $\mathcal{A}/\mathcal{G}$,

$$
\mathcal{X}(\mathcal{A}/\mathcal{G}) = \sum_{DF=0} sign \det ||\frac{\delta DF}{\delta A}||
$$

General arguments imply that this should coincide with (39). Indeed, by ignoring (irrelevant) complications that arise from a nontrivial moduli space we can show this directly, as a special case of the construction we have presented in the previous section. For this we simply apply our derivation of (97) to (80), using the explicit connection (38) and

the corresponding curvature two-form (36) (and ignoring inessential technical compli-
cations that arise since (38) is not a metric connection). In this way we immediately
find the Gauss-Bonnet-Chern representation of the Euler character, consistent with (16)
and (39). This also generalizes the original Atiyah-Jeffrey construction that we have
presented in section 3, to the case where we account for all solutions to the Yang-Mills
equation.

Here we are interested in the more general path integral (13),

$$Z_{YM} \approx \int [dQ][dA]\delta(DF) \det ||\frac{\delta DF}{\delta A}|| \exp\{-\int \frac{1}{4} F_{\mu\nu}^2 (A + Q)\}$$

which we interpret as an equivariant generalization of the Euler character, as we have
explained in section 2. Following our discussion in the previous section we introduce one
commuting variable π_μ and two anticommuting variables ψ_μ and $\bar{\mathcal{P}}_\mu$ and write (13) as

$$Z_{YM} \approx \int [dQ][dA][d\pi][d\psi][d\bar{\mathcal{P}}] \exp\{\int -\frac{1}{4} F_{\mu\nu}^2 (A + Q) + \pi_\nu D_\mu F_{\mu\nu} - \bar{\mathcal{P}}_\mu[F_{\mu\nu}, \psi_\nu]$$

$$- \frac{1}{2}(D_\mu\psi_\nu - D_\nu\psi_\mu)(D_\mu\bar{\mathcal{P}}_\nu - D_\nu\bar{\mathcal{P}}_\mu) \} \tag{98}$$

Here the first term depends on $A + Q$, while the remaining terms depend on A only.
Consequently we can again introduce the independent $\pm$ gauge fields (46) to conclude
that the action in (98) separates into a linear combination of an $A^+ \sim A + Q$ dependent
action described by the first term in (98), and an $A^- \sim A$ dependent action described
by the remaining terms in (98). As we have explained in section 5, this also means that
we have a local $SU(N)_+ \times SU(N)_-$ gauge symmetry, and the $\pm$ terms are coupled to each
other only by the requirement that the second Chern classes coincide,

$$\int F\tilde{F}(A^+) = \int F\tilde{F}(A^-)$$

Consider the A^- dependent, topological part of the action in (98),

$$S_{TOP}(A^-) = \pi_\nu D_\mu F_{\mu\nu} - \bar{\mathcal{P}}_\mu[F_{\mu\nu}, \psi_\nu] - \frac{1}{2}(D_\mu\psi_\nu - D_\nu\psi_\mu)(D_\mu\bar{\mathcal{P}}_\nu - D_\nu\bar{\mathcal{P}}_\mu) \tag{99}$$

As we have explained in the previous section, this action admits the following nilpotent
BRST (Parisi-Sourlas) symmetry

$$\Omega A_\mu^- = \psi_\mu$$

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$$\Omega\psi_\mu \;=\; 0$$
$$\Omega\bar{\mathcal{P}}_\mu \;=\; \pi_\mu$$
$$\Omega\pi_\mu \;=\; 0 \tag{100}$$

so that

$$\Omega \;=\; \psi_\mu E_\mu^- \;+\; \pi_\mu \bar{\eta}_\mu \tag{101}$$

where we have introduced the conjugate variable

$$\{\bar{\mathcal{P}}_\mu, \bar{\eta}_\nu\} \;=\; -\delta_{\mu\nu}$$

In particular, the action (99) can be represented as a BRST commutator,

$$- S_{TOP} \;=\; \{\Omega, \bar{\mathcal{P}}_\mu D_\nu F_{\mu\nu}\} \tag{102}$$

Furthermore, if we introduce the space components of the Parisi-Sourlas supergauge field

$$\mathcal{A}_\mu \;=\; A_\mu^- + \theta\psi_\mu - \bar{\theta}\bar{\mathcal{P}}_\mu + \theta\bar{\theta}\pi_\mu \tag{103}$$

we conclude from the previous section that we can write (99) as

$$S_{TOP}(\mathcal{A}) \;=\; \frac{1}{4}\int dx d\bar{\theta} d\theta \; \mathcal{F}_{\mu\nu}^2 \tag{104}$$

where $\mathcal{F}_{\mu\nu}$ denotes the space-time components of the Parisi-Sourlas field strength tensor,

$$\mathcal{F}_{\mu\nu} \;=\; \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{105}$$

Since Ω does not act on the A^+ field, the BRST symmetry (100) is also an invariance of the full action in (98). Consequently the path integral

$$Z_{YM} \;\approx\; \int [dQ][dA][d\pi][d\psi][d\bar{\mathcal{P}}] \exp\left\{\int -\frac{1}{4}F_{\mu\nu}^2(A^+) \;-\; \{\Omega, \Psi\}(A^-)\right\} \tag{106}$$

is invariant under local variations of the gauge fermion Ψ, and reproduces (98) when we select Ψ as in (102).

In analogy with the previous section, we now consider the following more general gauge fermion

$$\Psi \;=\; \bar{\mathcal{P}}_\mu(D_\nu F_{\mu\nu} + \kappa\pi_\mu) \tag{107}$$

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where $\kappa \propto m^2$ is a mass scale. Standard arguments imply that the ensuing path integral (106) is formally independent of κ, and when $\kappa \to 0$ we recover (102).

With nontrivial κ, the topological part of the action becomes

$$\{\Omega, \Psi\} = \pi_\nu D_\mu F_{\mu\nu} + \bar{\mathcal{P}}_\mu [F_{\mu\nu}, \psi_\nu] + \frac{1}{2}(D_\mu \psi_\nu - D_\nu \psi_\mu)(D_\mu \bar{\mathcal{P}}_\nu - D_\nu \bar{\mathcal{P}}_\mu) + \kappa \pi_\mu^2 \quad (108)$$

To interpret this, we introduce the full Parisi-Sourlas Yang-Mills field strength

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha + [\mathcal{A}_\alpha, \mathcal{A}_\beta]$$

where $\alpha, \beta = \mu, \theta, \bar{\theta}$. We then define the full Yang-Mills action in the Parisi-Sourlas superspace,

$$S_{PS} = \int d\theta d\bar{\theta} \, \frac{1}{4} \mathcal{F}^{\alpha\beta} \mathcal{F}_{\beta\alpha} \quad (109)$$

If we evaluate this action in the special case where

$$\mathcal{A}_\theta = \mathcal{A}_{\bar{\theta}} = 0$$

we find (108) by integrating over θ and $\bar{\theta}$. Hence we conclude that the topological action (99) in our path integral (98) determines a 4+2 dimensional Parisi-Sourlas Yang-Mills theory. Indeed, in the next section we shall argue, that $\mathcal{A}_\theta$ and $\mathcal{A}_{\bar{\theta}}$ can be identified with the ghosts that we need for a complete gauge fixing in the topological sector.

In analogy with (64) we can now break the $SU(N)_+ \times SU(N)_-$ gauge symmetry explicitly into the physical $SU(N)$ *e.g.* by adding an explicit mass term to the fluctuation field Q_μ. Indeed, a breaking of our extra symmetries becomes necessary if we want our construction to have nontrivial physical consequences: This gauge symmetry breaking also breaks the BRST symmetry in the topological sector and ensures that the κ dependence in (108) becomes nontrivial.

For this, following section 5 we introduce

$$S_{full} = \int \frac{1}{4} F_{\mu\nu}^2(A^+) + \frac{m^2}{2} Q_\mu^2 + \int \frac{1}{4} \mathcal{F}_{\alpha\beta}^2(\mathcal{A}) \quad (110)$$

As we have explained in section 5 unitarity is ensured by the horizontality condition

$$D_\mu Q_\mu = 0$$

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and since the mass term breaks the topological (Parisi-Sourlas) BRST invariance, the action (110) depends nontrivially also on the parameter κ which is necessary for physically nontrivial consequences.

It is interesting to consider our theory (110) in its topological sector: If we select the superspace analog of Feynman gauge, we find that the propagator of the superfield (103) is

$$< \mathcal{A}_\mu \mathcal{A}_\nu > = \delta_{\mu\nu} \left(\frac{1}{p^2} + \frac{\kappa \bar{\alpha} \alpha}{p^4} \right) \tag{111}$$

which exhibits the infrared $\mathcal{O}(p^{-4})$ behavior that leads to a linear potential between two static sources. This $\mathcal{O}(p^{-4})$ infrared behavior in (111) is unique in the following sense. By demanding locality and gauge invariance, we can generalize the gauge fermion in (107) by expanding it in derivatives of π_μ as follows,

$$\Psi = \bar{\mathcal{P}}_\mu (D_\nu F_{\mu\nu} + \kappa \pi_\mu + \kappa_1 D_\nu \pi_\nu \pi_\mu + \kappa_2 D^2 \pi_\mu + ...)$$

In the infrared limit $p \to 0$ we then conclude that the dominant contribution to the propagator indeed comes from (107).

The infrared behaviour of (111) suggests that in the topological sector our theory confines. Indeed, previously it has been conjectured [6] that in some sense the large distance limit of Yang-Mills vacuum can be viewed as a medium of randomly distributed color-electric and color-magnetic fields. This means that in the infrared limit Yang-Mills theory can be approximated by the following set of equations

$$\begin{aligned} D_\mu^{ab} F_{\mu\nu}^b &= h_\nu^a \\ < h_\mu^a(x) h_\nu^b(y) > &= \delta_{\mu\nu}^{ab} \end{aligned} \tag{112}$$

where the white noise random source h_μ^a describes the random color-electric and color-magnetic vacuum medium; Notice that since gauge invariance implies

$$D_\mu D_\nu F_{\mu\nu} = 0 \tag{113}$$

for consistency we must interpret the equations (112) to be defined on the gauge orbit $\mathcal{A}/\mathcal{G}$. The argument presented in [6] states, that as a consequence of the Parisi-Sourlas mechanism the equations (112) imply an effective dimensional reduction D=4 $\to$ D=2 in the infrared limit with the ensuing confinement of color. Indeed, by a direct computation

one can show that in the Parisi-Sourlas theory planar Wilson loops obey an area law, with string tension σ determined by κ

$$\sigma \;=\; \frac{1}{4\pi}\kappa N g^2$$

This is a direct consequence of the Parisi-Sourlas dimensional reduction, that relates (108) to the corresponding two-dimensional, ordinary Yang-Mills theory.

We observe, that the equations (112) are exactly those that we have derived earlier in (95), when we take into account the inessential complication (113) and interpret these equations on the gauge orbit $\mathcal{A}/\mathcal{G}$ and with a nontrivial moduli. Consequently we can view the present construction as a first principle derivation of the equations (112), suggesting that the qualitative picture developed in [6] might indeed be a proper way to describe Yang-Mills theories in the infrared limit. However, to ensure that the dependence on the parameter κ is nontrivial, in addition we need to break the BRST symmetry of the topological sector *e.g.* by adding a mass term as in (110).

9 Comparison of the Approaches

In the previous sections we have studied the Yang-Mills partition function in two different approaches. We have first investigated the instanton approximation (41), which yields an effective description based on the topological and the N=2 supersymmetric Yang-Mills theories. We have then considered the more general case (98), where we sum over all possible solutions to the classical Yang-Mills equation. The general arguments that we have presented in Section 2. suggest that these two approaches should essentially coincide. For this reason it is of interest to compare these approaches. In this way we can also expect to learn how good the instanton approximation actually is.

In addition of the conjugate Yang-Mills variables A_μ and E_μ we introduce the following pairs of canonically conjugated variables

$$\{\mathcal{X}_\mu, \psi_\nu\} \;=\; \{\bar{\mathcal{X}}_\mu, \bar{\psi}_\nu\} \;=\; \{\pi_\mu, \lambda_\nu\} \;=\; -\delta_{\mu\nu} \tag{114}$$

where we now use the notation in [8]. We combine our variables into the Parisi-Sourlas superfields as follows - see also (103)

$$\mathcal{A}_\mu(y) \;=\; A_\mu + \theta\psi_\mu + \bar{\theta}\bar{\mathcal{X}}_\mu - \theta\bar{\theta}\pi_\mu$$

$$\mathcal{E}_\mu(y) \quad = \quad \lambda_\mu + \theta\bar\psi_\mu + \mathcal{X}_\mu\bar\theta + \theta\bar\theta E_\mu \tag{115}$$

so that we have the superbrackets

$$\{\mathcal{E}_\mu(y_1), \mathcal{A}_\nu(y_2)\} \quad = \quad -\delta_{\mu\nu}(y_1 - y_2)$$

in the Parisi-Sourlas superspace. In order to construct the remaining θ, $\bar\theta$ components, we consider the nilpotent operator (101),

$$\Omega \quad = \quad \psi_\mu E_\mu \;+\; \pi_\mu\bar\psi_\mu \tag{116}$$

We identify this operator as the BRST operator for the constraint

$$E_\mu \;\approx\; 0 \tag{117}$$

However, these constraints (117) are not independent but are subject to Gauss law,

$$D_\mu E_\mu \;\approx\; 0 \tag{118}$$

which projects (116) to the gauge orbit space $\mathcal{A}/\mathcal{G}$. Such a linear relation among the constraints (117) means that the constraint algebra is reducible [17].

To implement Gauss law as a reducibility condition in the BRST operator (116), we introduce the following canonically conjugated Parisi-Sourlas superfields,

$$
\begin{aligned}
\mathcal{A}_\theta &= \eta - \theta\varphi - \bar\theta\lambda + \theta\bar\theta\bar\eta \\
\mathcal{E}_{\bar\theta} &= \bar{\mathcal{P}} + \theta\pi - \bar\theta p + \theta\bar\theta\mathcal{P}
\end{aligned}
\tag{119}
$$

and

$$
\begin{aligned}
\mathcal{A}_{\bar\theta} &= \bar b + \theta\bar\pi + \bar\theta b + \bar\theta\theta\rho \\
\mathcal{E}_\theta &= l + \theta c + \bar\theta\bar\lambda + \bar\theta\theta\bar c
\end{aligned}
\tag{120}
$$

where we use the notation in [8]. We introduce $g_{\mu\nu} = \delta_{\mu\nu}$ and $g_{\theta\bar\theta} = -g_{\bar\theta\theta} = 1$ and define the brackets of the various component fields in (119), (120) so that our superspace variables obey the following Poisson brackets

$$\{\mathcal{E}_\alpha(y_1), \mathcal{A}_\beta(y_2)\} \quad = \quad -g_{\alpha\beta}\delta(y_1 - y_2)$$

As explained in [8], the components of these additional superfields can be identified as the various ghosts that we need to introduce for a fully gauge fixed quantization of the constrained system (117), (118), according to the Batalin-Fradkin algorithm [17]. Consequently we generalize the BRST operator (116) into

$$\Omega \;\rightarrow\; \psi_\mu E_\mu + \bar\psi_\mu \pi_\mu + \varphi \mathcal{P} - \bar\eta\pi - c\rho + \bar{c}\bar\eta$$

$$= \int d\bar\theta d\theta \; g^{\alpha\beta}\partial_\theta \mathcal{A}_\alpha \mathcal{E}_\beta \;\sim\; \partial_\theta \tag{121}$$

which reproduces (86) on our variables.

If we now introduce the following conjugation of (121)

$$\Omega \;\rightarrow\; e^{-\Phi}\Omega e^{\Phi} \;=\; \Omega + \{\Omega, \Phi\} + \frac{1}{2!}\{\{\Omega, \Phi\}, \Phi\} \;+\; \dots$$

where we select

$$\Phi \;=\; \int d\bar\theta d\theta \; \theta(\mathcal{A}_\theta^a \mathcal{D}_i^{ab} \mathcal{E}_i^b + \frac{1}{2}f^{abc}\mathcal{A}_\theta^a \mathcal{A}_\theta^b \mathcal{E}_{\bar\theta}^c)$$

where the covariant derivative $\mathcal{D}_i^{ab}$ is with respect to the Parisi-Sourlas superconnection, we find for the conjugated Ω

$$\Omega \;=\; \int (g^{\alpha\beta}\partial_\theta \mathcal{A}_\alpha^a \mathcal{E}_\beta^a + \mathcal{A}_\theta^a \mathcal{D}_i^{ab} \mathcal{E}_\mu^b - \frac{1}{2}f^{abc}\mathcal{A}_\theta^a \mathcal{A}_\theta^b \mathcal{E}_{\bar\theta}^c) \tag{122}$$

Here the first term coincides with the translation operator (121) in the θ-direction, and the two remaining terms have the standard form of a nilpotent BRST operator for the superspace gauge transformation, with

$$\mathcal{G} \;=\; \mathcal{D}_\mu \mathcal{E}_\mu$$

the superspace Gauss law operator and $\mathcal{A}_\theta$, $\mathcal{E}_{\bar\theta}$ viewed as the superspace ghost.

We are now in a position to compare the anti-selfdual instanton approximation (41) to (98) which accounts for all possible solutions to the classical Yang-Mills equation. For this, we substitute our component field expansions (115), (119) and (120) in the BRST operator (122). Comparing with the BRST operator (25) of topological Yang-Mills theory we then find that the functional forms of (122) and (25) coincide, term-by-term. The only difference between these two operators comes from the anti-selfduality of the two-form variables in (25), which have been replaced by vectors in (122): The anti-selfdual tensor has three independent components, while a vector has four. Consequently

this replacement means an extension of the pertinent complex. Such an extension is natural, since the full Yang-Mills equation contains both the anti-selfdual and selfdual equations. The extenxion then simply reflects the fact that while (122) accounts for all possible solutions to the Yang-Mills equation, in (25) we only consider the anti-selfdual ones. Indeed, a restriction of (122) to the anti-selfdual sector yields immediately the BRST operator (25).

From the preceding discussion we conclude that our two approaches (12) and (17) *essentially coincide*: Except for the selfduality condition, the only additional difference arises from the choice of the gauge fermion Ψ. However, due to the Ψ-independence of the partition function this difference is entirely irrelevant.

Since the topological Yang-Mills theory is related to the N=2 supersymmetric Yang-Mills theory by the invertible change of variables (71), the Parisi-Sourlas Yang-Mills theory must also contain the N=2 theory, *both* in its selfdual and antiselfdual subsectors. This means in particular, that the confinement mechanism which has been identified in the N=2 theory [1] is also contained in the topological Parisi-Sourlas sector of ordinary Yang-Mills theory. Thus there should be a direct relation between the picture of confinement by monopole condensation developed in [2], [1] and the picture of confinement by randomly fluctuating color-electric and color-magnetic fields developed in [6]. Furthermore, the derivation [7] that planar Wilson loops in the Parisi-Sourlas theory obey an area law should also be directly applicable to the corresponding Wilson loops in the N=2 approach, modulo a change of variables that originates from the different choice of gauge fermions Ψ.

Finally, in our discussion of (41) and (98) we have ignored the phase factors that appear in (20) and (21). However, we do not expect these phase factors to modify the qualitative aspects of our results: Since these phase factors depend only on the topological connection A^-, they do not break the $SU_+(N) \times SU_-(N)$ gauge symmetries. However, they do explicitly break the BRST supersymmetries, and in particular the Ψ independence in the topological sectors, but in a controllable fashion. Since the BRST transformation is a change of variables of the form

$$\phi^a \;\rightarrow\; \phi^a \;+\; \delta\Psi\{\Omega, \phi^a\}$$

we conclude that when we vary the gauge fermion $\Psi \rightarrow \Psi + \delta\Psi$ the phase factors do

not remain intact but suffer a nontrivial change of variables. Besides the terms that we have discussed in the previous sections, we should then add the phase factors which have been subjected to the proper changes of variables. These are additional non-local terms that should be included to our action. However, since these terms are additive and highly complicated, we can use general arguments to conjecture that they become irrelevant operators in the infrared limit.

10   Conclusions

In conclusion, by applying background field formalism in the path integral approach we have introduced new variables to describe ordinary Yang-Mills theory. Using our new variables, we have then established that in the instanton approximation ordinary Yang-Mills theory *contains* the N=2 supersymmetric Yang-Mills theory. Furthermore, by considering all possible solutions to the Yang-Mills equations, we have established that this N=2 supersymmetry is embedded in the Parisi-Sourlas supersymmetry. Finally, we have found that the SU(N) gauge symmetry of our Yang-Mills theory becomes extended into an SU(N)×SU(N) gauge symmetry. Such extensions of the original Yang-Mills symmetry opens the interesting possibility to explicitly break the extra symmetries back into the original SU(N) gauge invariance. We have investigated in detail a particular explicit breaking of the additional symmetries, with intriguing physical implications. Our symmetry breaking emerges, when we add a mass term to the field that describes quantum fluctuations around the classical solutions in the background field formalism. We have argued that this mass term is consistent both with power counting renormalizability and unitarity. In this way we then obtain an extension of the original Yang-Mills theory, with a gauge invariant mass scale. We have argued that our construction suggests an approach to color confinement by the Parisi-Sourlas dimensional reduction mechanism. In this picture, the large distance fluctuations of the Yang-Mills vacuum can be viewed as a medium of randomly distributed color-electric and color-magnetic fields. We have derived the relevant equations of motion from a first principle. Since the Parisi-Sourlas extended Yang-Mills theory contains the N=2 supersymmetric Yang-Mills theory in its selfdual and anti-selfdual sectors, we have also concluded that this approach to confinement in ordinary Yang-Mills theory coincides with the recent proposal to describe

confinement in N=2 supersymmetric theories. In particular, the area law for Wilson loops derived in the Parisi-Sourlas theory should immediately extend to the N=2 supersymmetric case.

Finally, we point out that recently Polyakov [20] has proposed a string version of Yang-Mills theory, that also exhibits Parisi-Sourlas supersymmetry. We view this as a further indication, that the Parisi-Sourlas dimensional reduction is indeed the underlying reason for color and quark confinement in ordinary Yang-Mills theory. Indeed, the 4+2 dimensional Parisi-Sourlas supersymmetric Yang-Mills theory is intimately connected with the two dimensional ordinary Yang-Mills theory and the latter admits a natural string interpretation [21]. This suggests, that there should also be an intimate relationship between the string theory constructed by Polyakov and the string theory that describes the two dimensional Yang-Mills theory. The understanding of this relationship might provide an important clue for constructing the string variables of four dimensional Yang-Mills theory.

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